Chapter 6: Small Oscillations
Part 1

Application: Small oscillations has applications in acoustics, molecular spectra, vibrations of mechanisms, coupled electrical circuits, and many other areas in physics.

Overview
- Formulating the problem
- Proof of eigenvalues that are real and positive definite
- Normalization of eigenvalues
- Algorithm for solving small oscillations problems

Let $L = L(q_i, \dot{q}_i, t)$ satisfy Lagrange's equations. Then $L'(q_i, \dot{q}_i, t) + \frac{dF}{dt}$ also satisfies Lagrange's equations, with $F = F(q_i, t)$

Also, $T = \sum_i \frac{m_i}{2} \bar{v}_i^{\,2} = \sum_i \frac{m_i}{2} \left(\frac{d\bar{r}_i}{dt} + \sum_j \frac{d\bar{r}}{dq_j} \frac{dq_j}{dt} \right)^2$ (1.71)'

Expanding, we get 3 terms. Call them: T_0, T_1, T_2

$T_0 = \sum_i \frac{m_i}{2} \left(\frac{d\bar{r}_i}{dt} \right)^2 \rightarrow$ independent of generalized velocities

$T_1 = \sum_{i,j} m_i \left(\frac{d\bar{r}_i}{dt} \right) \left(\frac{d\bar{r}_i}{dq_j} \right) \dot{q}_j \rightarrow$ linear with generalized velocities

$T_2 = \sum_{i,j,k} \frac{m_i}{2} \left(\frac{d\bar{r}_i}{dq_j} \right) \left(\frac{d\bar{r}_i}{dq_k} \right) \dot{q}_j \dot{q}_k \rightarrow$ quadratic with generalized velocities

with $T = T_0 + T_1 + T_2$ (1.73)

Note: if $\bar{r}_i$ has no explicit time dependence,

$\frac{d\bar{r}_i}{dt} = 0 \Rightarrow T_0 = T_1 = 0 \Rightarrow T = T_2$

For oscillations, assume generalized coordinates with no time dependence. We consider conservative systems with potentials that are functions of position only.

A system is in equilibrium if generalized forces on the system are zero

$$Q_i = -\left(\frac{\partial V}{\partial q_i}\right)_0 = 0$$

Consider the neighborhood of the minimum potential $V(q_i)$ at the point $\bar{q}_0 \equiv \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}$

Expanded with a Taylor Series,

$$\Rightarrow V(q_i) = V(q_{01}, q_{02}, \dots q_{0n}) + \sum_i \left(\frac{\partial V}{\partial q_i}\right)_{\bar{q}_0} (\bar{q} - \bar{q}_0)_i$$

$$+ \frac{1}{2} \sum_i \sum_j \frac{\partial^2 V}{\partial q_i \partial q_j} (\bar{q} - \bar{q}_0)_i (\bar{q} - \bar{q}_0)_j + \dots$$

By choosing a small enough neighborhood of V, we can neglect the higher order terms of the expansion.

Also, by redefining our origin to coincide with the equilibrium potential, the first term vanishes. So, after recalling that $\frac{\partial V}{\partial q_i} = 0$, we are left with

$$V(q_i) = \frac{1}{2} \sum_i \sum_j \frac{\partial^2 V}{\partial q_i \partial q_j} (\bar{q} - \bar{q}_0)_i (\bar{q} - \bar{q}_0)_j \qquad (6.4)$$

From (1.71), $T = \sum_{i,j} \frac{m_{ij}}{2} \dot{q}_i \dot{q}_j$, with $m_{ij} = m_{ij}(q_i)$ in general

but taking the Taylor Expansion and keeping the lowest order term, $m_{ij}(\bar{q}) = m_{ij}(\bar{q}_0)$

Let then $m_{ij}(\bar{q}_0) \equiv T_{ij}$

$$\Rightarrow T = \frac{1}{2} \sum_i \sum_j \dot{q}_i T_{ij} \dot{q}_j = \frac{1}{2} \dot{\bar{q}}^T \bar{\bar{T}} \dot{\bar{q}}$$

$$\Rightarrow L = T - V = \frac{1}{2} \left(\dot{\bar{q}}_n^T \bar{\bar{T}} \dot{\bar{q}}_n - \bar{q}_n^T V q_n \right) \quad \text{with } \bar{q}_n = (\bar{q} - \bar{q}_0)$$

and $\dot{\bar{q}}_n = \dot{\bar{q}}$ Let $\bar{q}_0 = 0$ to drop the n subscript (6.7)

By Lagrange's equations: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\bar{q}}}\right) - \frac{\partial L}{\partial \bar{q}} = 0$

And we get our equations of motion

$$\bar{\bar{T}}\ddot{\bar{q}} + \bar{\bar{V}}\bar{q} = 0 \qquad (6.8)$$

$$\Rightarrow \bar{\bar{T}}\ddot{\bar{q}} = -\bar{\bar{V}}\bar{q} \quad \Rightarrow \quad \ddot{\bar{q}} = -\bar{\bar{T}}^{-1}\bar{\bar{V}}\bar{q}$$