Final Examination for PHYS 6220/7220, Fall 2009

1. A particle moves in a central potential $V(r) = (kr^2)/2$. The constant k is positive and has appropriate dimensions. The particle has mass m, energy E, and angular momentum ℓ about the center of force. We need to analyze this potential for the nature of the motion of the particle in detail so that the problem is completely solved. We will need to accomplish the following tasks for this purpose.

(i) Write the Lagrangian for the system.
(ii) Eliminate any variable we can and reduce the Euler-Lagrange equations to that of a one dimensional problem.
(iii) Write an expression that may be further analyzed for the nature of the motion. A condition that some physical quantity be real can be imposed from this equation. State that condition.
(iv) From answers in part (iii) note the nature of orbits for various values of ℓ and E. In doing so we can follow these hints: Draw the effective potential $V_e = V_e(r)$. If they exist state forbidden values of E. State the conditions for circular orbits to occur. Find the radius of these circular orbits if they exist. State the limits on the radius for bounded orbits to occur, if they are possible. State if and under what conditions unbounded orbits can occur.
(v) Write the differential and integral equation for the orbit equation.
(vi) Solve any of these equations if you can. From these draw with properly marked X, Y axis and the center of force all possible bounded and unbounded orbits. Mark all turning points and any other special points if they exist.
(vii) State (do not solve) from part (vi) how you would get r(t) and θ(t).
(viii) State any other analysis, checks, heuristic arguments that need to be performed to complete this problem and have not been considered above.

(a) Perform the tasks (i) – (iii). (**3 points**)
(b) For the case $\ell = 0$ perform the tasks (iv) – (vii) in detail. (**3 points**)
(c) For the case $\ell > 0$ perform the tasks from (iv) – (vii) in detail. (**9 points**)
(d) Perform task (viii). (**2 points**)

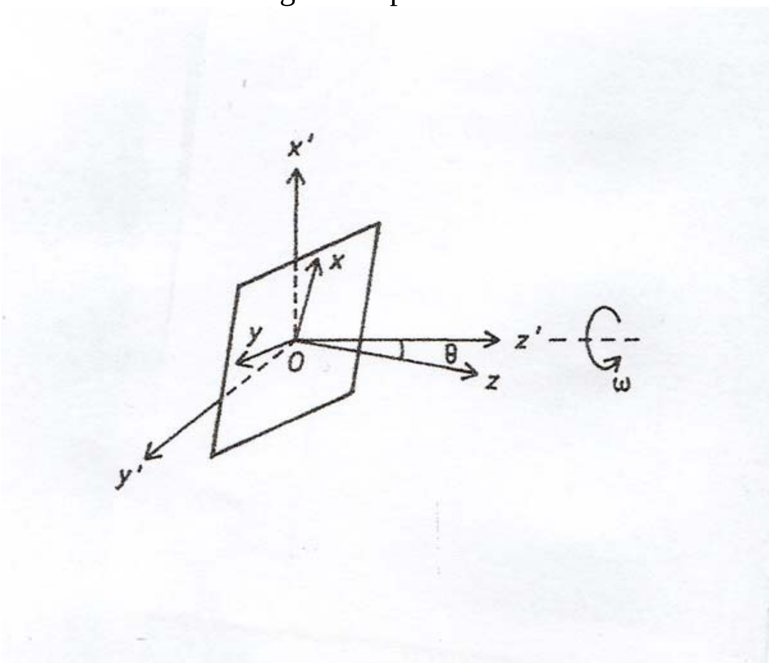
2. A thin square plate of mass m, with side of length b rotates at a constant angular frequency ω about an axis through the center of the plate tilted by an angle θ with respect to the normal to the plate as shown in the figure. The body axes are denoted by xyz and the laboratory axis by x'y'z'. The common origin O of both systems is at the center of the plate. The angle θ remains constant at all times. The z axis is normal to the plate. The x and y axes are chosen along the sides of the square such that the three axes x, z' and z are coplanar at all times.
(a) Compute the moment of inertia matrix of the plate in the body frame. Comment on your result. (**4 points**)
(b) Express the angular velocity of rotation in the laboratory frame and the body frame. (**1 point**)
(c) Compute the angular momentum of the plate in the body frame at all times. Compute the torque on the plate in the body frame. (**2 points**)

(d) Compute the rotation matrix that transforms the unit vectors **i', j', k'** to **i, j, k** at all times. You may assume that the y and y' axis coincide at t = 0. Hints: (i) Axes x, z', and z are always coplanar, (ii) Both systems are right handed and orthogonal. (iii) These are all unit vectors. (v) Components of **i, j, k** parallel and perpendicular to vector **ω** behave in simple ways as a function of time. (vi) Vector **ω** is parallel to one of the six vectors. (**6 points**)

(e) Compute the angular momentum of the plate in the laboratory frame from results in parts (c) and (d). (**3 points**)

(f) Compute the torque on the plate in the laboratory frame. (**1 point**)

Figure for problem 2



Some relevant and irrelevant information for both problems.

$\int \frac{xdx}{\left(-a+bx^2-x^4\right)^{1/2}} = \left(\frac{1}{2}\right)\arcsin\left[\frac{2x^2-b}{\sqrt{b^2-4a}}\right]+C;$ where C is a constant of integration, and the integrand on the left is always real.