For a central force problem we have

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$$\bar{F}(\bar{r}) = f(r)\frac{\bar{r}}{r}$$

$$\therefore \quad \dot{\bar{p}} = f(r)\frac{\bar{r}}{r}$$

$$\bar{L} \equiv \bar{r}\times\bar{p} = \text{constant}$$

$$\therefore \frac{d}{dt}(\bar{p}\times\bar{L}) = \dot{\bar{p}}\times\bar{L} = f(r)\,\frac{\bar{r}}{r}\times\bar{L}$$

$$= m\frac{f(r)}{r}\left[\bar{r}\times(\bar{r}\times\dot{\bar{r}})\right]$$

$$= \frac{mf(r)}{r}\left[\bar{r}(\bar{r}\cdot\dot{\bar{r}}) - (\bar{r}\cdot\bar{r})\dot{\bar{r}}\right]$$

where we used $\bar{A}\times(\bar{B}\times\bar{C}) = \bar{B}(\bar{A}\cdot\bar{C}) - \bar{C}(\bar{A}\cdot\bar{B})$

$$\therefore \frac{d}{dt}(\bar{p}\times\bar{L}) = m\frac{f(r)}{r}\left[\bar{r}\frac{d}{dt}\left(\frac{\bar{r}\cdot\bar{r}}{2}\right) - r^2\dot{\bar{r}}\right]$$

$$= m\frac{f(r)}{r}\left[\bar{r}r\dot{r} - r^2\dot{\bar{r}}\right]$$

$$= mf(r)\left[\bar{r}\dot{r} - r\dot{\bar{r}}\right] = \frac{mf(r)r^2}{r^2}\left[\bar{r}\dot{r} - r\dot{\bar{r}}\right]$$

$$= -mf(r)r^2\left[\frac{r\dot{\bar{r}} - \bar{r}\dot{r}}{r^2}\right] = -mf(r)r^2\frac{d}{dt}\left(\frac{\bar{r}}{r}\right)$$

$$\therefore \frac{d}{dt}(\bar{p}\times\bar{L}) + mf(r)r^2\frac{d}{dt}\left(\frac{\bar{r}}{r}\right) = 0$$

If $V(r) = -\frac{k}{r} \Rightarrow f(r) = -\frac{\partial V}{\partial r} = -\frac{k}{r^2}$

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$$\therefore \quad \frac{d}{dt}\left[\bar{p} \times \bar{L} - mk\frac{\bar{r}}{r}\right] = 0$$

$$\Rightarrow \quad \bar{p} \times \bar{L} - mk\frac{\bar{r}}{r} = \bar{A} = \text{constant vector}$$

$\bar{A}$ is called the Laplace-Runge-Lenz vector. Since $\bar{A}$ is constant let us calculate it at the perihelion

$$r = \frac{a(1-\epsilon^2)}{1+\epsilon\cos\theta}$$

$$r_{min} = a(1-\epsilon)$$

$$\bar{p} = m(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})$$

at $r = r_{min}$, $\dot{r} = 0$

$$\Rightarrow \bar{p} = mr\dot{\theta}\hat{\theta}$$

$\bar{L} = l\hat{z}$ = normal to the plane

$$\Rightarrow \quad \bar{p} \times \bar{L} = mrl\dot{\theta}(\hat{\theta} \times \hat{z})$$

$$= ml r_{min}\dot{\theta}\hat{r}$$

$$\therefore |\bar{A}| = |\bar{p} \times \bar{L} - mk\hat{r}| = m[l r_{min}\dot{\theta} - k]$$

But $\dot{\theta} = l/(mr^2)$

$$l r_{min} \dot{\theta} = l^2/(m r_{min}) = l^2/[ma(1-\epsilon)]$$

Also we have

$$\frac{l^2}{mk} = a(1-\epsilon^2)$$

$$\therefore \frac{l^2}{ma(1-\epsilon)} = k[1+\epsilon]$$

$$\therefore |\bar{A}| = mk\epsilon$$

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Also $\epsilon = \sqrt{1 + \frac{2El^2}{mk^2}}$

$$\Rightarrow |\bar{A}|^2 = m^2k^2 + 2mEl^2 \longrightarrow 3.87$$

Note also that $\exists$ 7 constants we have discovered $\bar{A}$, $\bar{L}$ and E since $\bar{A}$ & $\bar{L}$ are vectors.

$\bar{L}\cdot(\bar{p}\times\bar{L}) = 0$ also $\bar{L}\cdot\bar{r} = (\bar{r}\times\bar{p})\cdot\bar{r} = 0$

$$\Rightarrow \bar{L}\cdot\bar{A} = 0 \longrightarrow 3.83$$

With the constraints 3.83 and 3.87 there are only 5 independent constants.

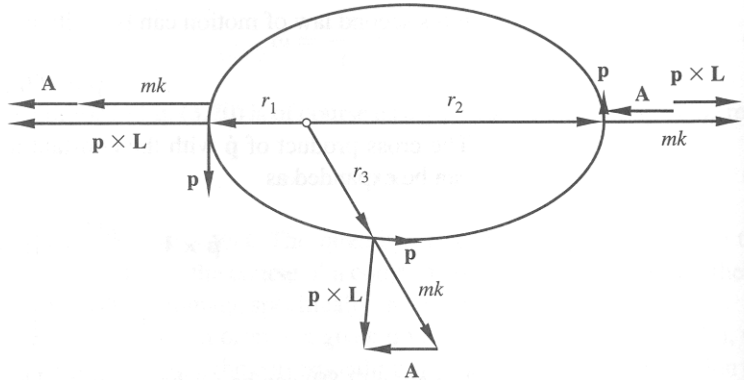


FIGURE 3.18 The vectors **p**, **L**, and **A** at three positions in a Keplerian orbit. At perihelion (extreme left) $|\mathbf{p}\times\mathbf{L}| = mk(1+e)$ and at aphelion (extreme right) $|\mathbf{p}\times\mathbf{L}| = mk(1-e)$. The vector **A** always points in the same direction with a magnitude mke.

the orbit. If θ is used to denote the angle between **r** and the fixed direction of **A**, then the dot product of **r** and **A** is given by

$$\mathbf{A}\cdot\mathbf{r} = Ar\cos\theta = \mathbf{r}\cdot(\mathbf{p}\times\mathbf{L}) - mkr. \tag{3.84}$$

Now, by permutation of the terms in the triple dot product, we have

$$\mathbf{r}\cdot(\mathbf{p}\times\mathbf{L}) = \mathbf{L}\cdot(\mathbf{r}\times\mathbf{p}) = l^2,$$

so that Eq. (3.84) becomes

$$Ar\cos\theta = l^2 - mkr,$$

or

$$\frac{1}{r} = \frac{mk}{l^2}\left(1 + \frac{A}{mk}\cos\theta\right). \tag{3.85}$$

Scattering in a central force

The differential cross section

$$d\sigma_R(\bar{\Omega}) \equiv \sigma_R(\bar{\Omega})\,d\Omega = \frac{N_R(\bar{\Omega})}{I}$$

where I = incident intensity

$N_R(\bar{\Omega})$ = number of particles scattered into the solid angle ($d\Omega$) per unit time.

$$[I] = [M^0 L^{-2} T^{-1}]$$

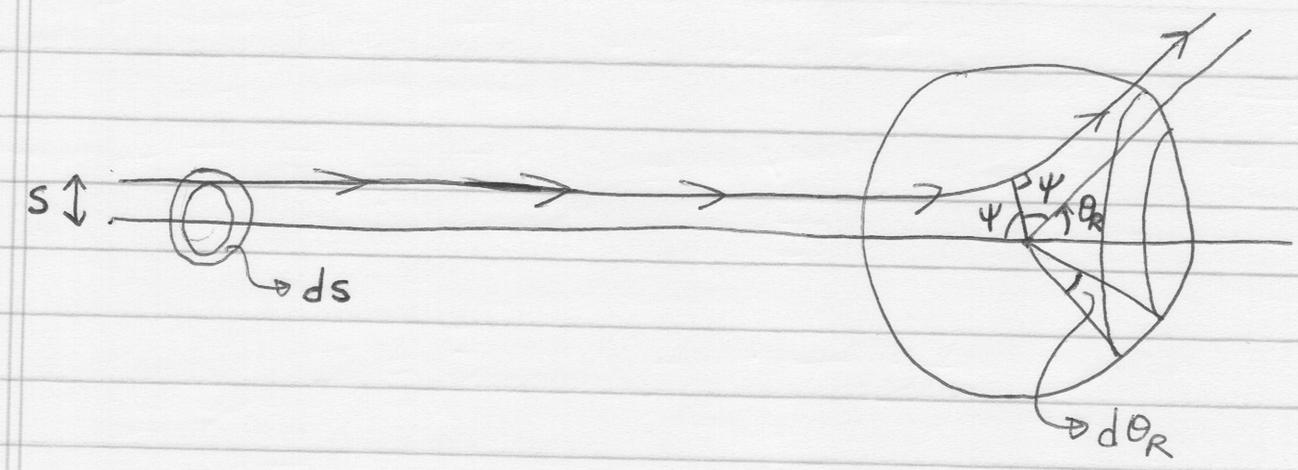
$$[N_R(\bar{\Omega})] = [M^0 L^0 T^{-1}]$$

$$[d\sigma_R(\bar{\Omega})] = [M^0 L^2 T^0]$$

In central force problems $\exists$ symmetry around the axis of the incident beam which is assumed to pass through the center of force.

$$\therefore \quad d\Omega = 2\pi \sin\theta_R \, d\theta_R$$

where θ_R = angle between the scattered and incident directions.



Let v_0 = initial speed of the incident particles, and s = impact parameter.

s = distance from the center of force of the line determined by the initial particle position and velocity.

$\therefore \ |\vec{L}| = l = m v_0 s = s\sqrt{2mE} \longrightarrow$ (3.90)

since $2E = m v_0^2$ = initial energy $\times 2$.

Assume for simplicity that s determines θ_R uniquely.

$$\Rightarrow \quad 2\pi I s|ds| = 2\pi\, \sigma_R(\theta_R)\, I \sin\theta_R |d\theta_R|$$

where we denote $\sigma_R(\theta_R) \equiv \sigma_R(\Omega)$ for central forces.

$$\Rightarrow \quad \sigma_R(\theta_R) = \frac{s}{\sin(\theta_R)}\left|\frac{ds}{d\theta_R}\right| \longrightarrow \text{(3.93)}$$

From figure we see that

$$\theta_R + 2\Psi = \pi \quad \longrightarrow (3.94)$$

where Ψ is the angle between the incident velocity $\bar{v}_0$ and the line from the center of force to the point of closest approach. The latter is called the periapsis.

Eq. (3.36) reads

$$\theta = \int_{r_0}^{r} \frac{dr}{\sqrt{\dfrac{2mEr^4}{l^2} - \dfrac{2mVr^4}{l^2} - r^2}} + \theta_0 \quad \longrightarrow (3.36)$$

At the initial point $r_0 = \infty$ and $\theta_0 = \pi$

Also $\quad \theta - \Psi = \theta_R \quad \longrightarrow (3.945)$

(3.945) with (3.94) $\Rightarrow \Psi = \pi - \theta$,

which with (3.36) gives

$$\Psi = \int_{r_m}^{\infty} \frac{l\,dr}{\sqrt{2mEr^4 - 2mVr^4 - r^2 l^2}}$$

$$\therefore \theta_R = \pi - 2\Psi = \pi - 2\int_{r_m}^{\infty} \frac{s\,dr}{\left[r^4\left(1 - \frac{V(r)}{E}\right) - s^2 r^2\right]^{1/2}}$$

Let $u = 1/r$

$$\Rightarrow \theta_R(s) = \pi - 2\int_0^{u_m} \frac{s\,du}{\sqrt{1 - \frac{V(1/u)}{E} - s^2u^2}} \longrightarrow (3.97)$$

Consider now the center of force to have a charge $-Ze$, e = proton charge. Let the scattering particle have charge $-Z'e$, $Z > 0$, $Z' > 0$

$$\Rightarrow \text{force } f = \frac{ZZ'e^2}{r^2}$$

$$\Rightarrow V(r) = \frac{ZZ'e^2}{r} \equiv -k/r, \quad k = -ZZ'e^2 \longrightarrow (3.98)$$

Note $k < 0 \Rightarrow$ repulsive potential.

$$E = \frac{mV_0^2}{2} > 0 \Rightarrow \epsilon > 1$$

By (3.57) $\epsilon = \sqrt{1 + \frac{2El^2}{mk^2}}$

which using (3.98) and (3.90) becomes

$$\epsilon = \sqrt{1 + \left(\frac{2Es}{ZZ'e^2}\right)^2} \longrightarrow (3.99)$$

Eq. 3.55 gives

$$r = \frac{l^2}{mk\left[1+\epsilon\cos(\theta-\theta_0)\right]}$$

$$r \to \infty \Rightarrow \cos(\theta-\theta_0) = -1/\epsilon$$

Let $\theta - \theta_0 = \pi - \psi$

$$\Rightarrow \cos\psi = 1/\epsilon$$

$$\theta_R = \pi - 2\psi \Rightarrow \psi = \left(\frac{\pi-\theta_R}{2}\right)$$

$$\Rightarrow \cos\left(\frac{\pi}{2}-\frac{\theta_R}{2}\right) = \sin\left(\frac{\theta_R}{2}\right) = 1/\epsilon$$

$$\Rightarrow \cot^2\left(\frac{\theta_R}{2}\right) = \operatorname{cosec}^2\left(\frac{\theta_R}{2}\right) - 1 = \epsilon^2 - 1$$

$$= \frac{2Es}{ZZ'e^2}$$

$$\Rightarrow s = \frac{ZZ'e^2}{2E}\cot\left(\frac{\theta_R}{2}\right)$$

By 3.93

$$\sigma_R(\theta_R) = \frac{1}{4}\left(\frac{ZZ'e^2}{2E}\right)^2 \operatorname{cosec}^4\left(\frac{\theta_R}{2}\right)$$

The total cross-section

$$\sigma_{RT} = \int_{4\pi} \sigma_R(\Omega)\, d\Omega = 2\pi\int_0^\pi \sigma_R(\theta_R)\sin(\theta_R)\, d\theta_R$$

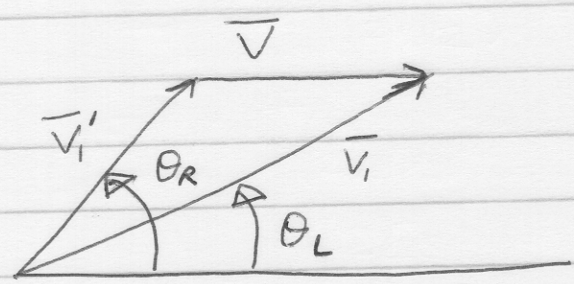
$\sigma_{RT} \to \infty$ for Coulomb scattering.

Scattering in the laboratory frame: →

Let $\bar{V}_0$ = initial velocity of m_1
Assume m_2 is at rest initially
Let $\bar{V}_1$ = final velocity of m_1.

$$\therefore \quad m_1 \bar{V}_0 = (m_1 + m_2)\,\bar{V}$$

$\bar{V} \equiv$ COM velocity



From figure above

$$V_1 \sin\theta_L = V_1' \sin\theta_R$$

$$V_1 \cos\theta_L = V_1' \cos\theta_R + V$$

$$\Rightarrow \tan\theta_L = \frac{\sin\theta_R}{\cos\theta_R + \rho}$$

$$\rho = V_1'/V = \frac{V_1'(m_1+m_2)}{V_0 m_1}$$

$$\cos\theta_L = (1+\tan^2\theta_L)^{-1/2} = \frac{\rho + \cos\theta_R}{\sqrt{1+\rho^2+2\rho\cos\theta_R}} \quad \text{3.110}$$

Also

$$\bar{V}_1' = \bar{V}_1 - \bar{V} = \bar{V}_1 - \frac{m_1\bar{V}_1 + m_2\bar{V}_2}{m_1+m_2}$$

$$= \frac{m_2\bar{V}_{12}}{m_1+m_2}, \quad \text{where } \bar{V}_{12} = \bar{V}_1 - \bar{V}_2$$

and $\bar{V}_2$ = velocity of the second particle after collision,

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$$\therefore |\bar{V}_1'| \equiv V_1' = \frac{m_2}{m_1+m_2} V_{12},$$

$$\therefore \rho = \frac{m_1 V_0}{m_2 V_{12}}$$

Let φ of the collision be defined by

$$\varphi = \frac{1}{2}\left(\frac{m_1 m_2}{m_1+m_2}\right)\left(V_{12}^2 - V_0^2\right)$$

$$= \frac{1}{2}\left(\frac{m_1 V_0^2 m_2}{m_1+m_2}\right)\left[\left(\frac{V_{12}}{V_0}\right)^2 - 1\right], \quad \text{but } E = \frac{m_1 V_0^2}{2}$$

$$\therefore \frac{V_{12}}{V_0} = \sqrt{1 + \frac{(m_1+m_2)\varphi}{m_2 E}}$$

$$\therefore \rho = \frac{m_1 E^{1/2}}{\left[m_2^2(E+\varphi) + m_1 m_2 \varphi\right]^{1/2}} \longrightarrow 3.114$$

For elastic scattering there is no energy loss $\Longleftrightarrow \varphi = 0$

$$\Rightarrow \quad \rho = m_1/m_2 \longrightarrow (3.111)$$

Let $d\sigma_L(\theta_L) \equiv \sigma_L(\theta_L)|d\theta_L|\, 2\pi \sin(\theta_L)$

be the differential cross-section as measured in the laboratory, with θ_L as argument.

Conservation of particles

$$\Rightarrow \quad 2\pi I \sigma_R \sin(\theta_R)|d\theta_R| = 2\pi I \sigma_L(\theta_L) \sin\theta_L |d\theta_L|$$

$$\Rightarrow \quad \sigma_L(\theta_L) = \sigma(\theta_R) \left| \frac{d(\cos\theta_R)}{d(\cos\theta_L} \right| \longrightarrow (3.115)$$

Using (3.110) in (3.115) we get

$$\sigma_L(\theta_L) = \sigma(\theta_R) \frac{\left[1 + 2\rho\cos\theta_R + \rho^2\right]^{3/2}}{1 + \rho\cos\theta_R}$$

Let $E_1 = \frac{m_1 v_1^2}{2}$ = final energy of the incident particle

$$\therefore \quad E_1/E = \frac{v_1^2}{v_0^2} = \frac{(\bar{v}_1' + \bar{V})^2}{v_0^2} = \frac{v_1'^2 + V^2 + 2v_1' V \cos\theta_R}{v_0^2}$$

$$= \frac{v_1'^2}{v_0^2} + \frac{V^2}{v_0^2} + \frac{2v_1'}{v_0}\cos\theta_R = \left(\frac{m_1}{m_1+m_2}\right)^2 \left[\frac{1+\rho^2+2\rho\cos\theta_R}{\rho^2}\right]$$