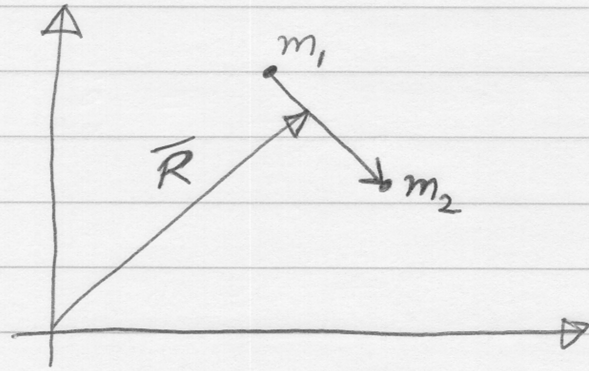
Central Forces :→

The 2-body problem



Consider only a 2 particle system

$$L = T - U(\bar{r}_1, \bar{r}_2, \dot{\bar{r}}_1, \dot{\bar{r}}_2, t)$$

Let $T = \frac{m_1}{2}\dot{\bar{r}}_1^{\,2} + \frac{m_2}{2}\dot{\bar{r}}_2^{\,2}$

For COM we get $\bar{R}(m_1+m_2) = m_1\bar{r}_1 + m_2\bar{r}_2$

Let $\bar{r} \equiv \bar{r}_2 - \bar{r}_1$

$$\Rightarrow \bar{R} = \frac{m_1\bar{r}_1 + m_2(\bar{r}+\bar{r}_1)}{m_1+m_2} = \bar{r}_1 + \frac{m_2\bar{r}}{m_1+m_2}$$

$$\Rightarrow m_1\bar{r}_1 = m_1\bar{R} - \frac{m_1 m_2\bar{r}}{m_1+m_2}$$

$$\Rightarrow m_1\dot{\bar{r}}_1 = m_1\dot{\bar{R}} - \frac{m_1 m_2}{m_1+m_2}\dot{\bar{r}}$$

Also $m_2\dot{\bar{r}}_2 = m_2\dot{\bar{R}} + \frac{m_1 m_2\dot{\bar{r}}}{m_1+m_2}$

$$\Rightarrow T = \frac{(m_1+m_2)}{2}\dot{\bar{R}}^2 + \frac{1}{2}\left(\frac{m_1 m_2}{m_1+m_2}\right)\dot{\bar{r}}^2$$

$\Rightarrow$ $T = \frac{m_1+m_2}{2}\dot{\bar{R}}^2 + \frac{\mu}{2}\dot{\bar{r}}^2$

where $\mu = \frac{m_1 m_2}{m_1+m_2}$ = reduced mass

Now let $U(\{\bar{r}_i\},\{\dot{\bar{r}}_i\},t) = V(\{(\bar{r}_1 - \bar{r}_2)\})$

We sit in the center of mass (COM) frame. Then $\bar{R}=0$, $\dot{\bar{R}}=0$.

$\therefore L = \frac{\mu}{2}\dot{\bar{r}}^2 - V(r)$

Henceforth let $m \equiv \mu$.
Now the problem has been reduced to that of single particle.

Since $V = V(r)$ and $\frac{\partial L}{\partial t} = 0$

$\dot{H} = 0$, $H = T + V = \frac{m}{2}\dot{\bar{r}}^2 + V(r)$

$\bar{L} \equiv \bar{r}\times\bar{p} \equiv m\bar{r}\times\dot{\bar{r}}$

$\Rightarrow$ $\frac{\partial \bar{L}}{\partial t} = 0$

$\Rightarrow$ $\frac{d\bar{L}}{dt} = \dot{\bar{L}} = [\bar{L}, H] = 0$, since

V is a function of r only
i.e. it is spherically symmetric.

Prove $[\bar{L}, H] = 0$ as an excercise.

Since $\bar{L}$ = constant vector we get

$$\bar{r} \times \bar{p} = \text{constant vector}$$

Let us choose our co-ordinate axes $\ni$ $\bar{L} = l\hat{z}$

$$\Rightarrow \bar{r}\cdot\bar{L} = 0 \text{ since}$$

$$\bar{L} = \bar{r} \times \bar{p} \text{ is } \perp \text{ to } \bar{r}$$

$$\Rightarrow \bar{r}\cdot\hat{z} = 0.$$

Let us choose the origin $\ni$ $\bar{r}$ lies in the XY plane.

$$\Rightarrow \bar{L} = mr\,\hat{r} \times (\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) = l\hat{z}$$

$$\Rightarrow mr^2\dot{\theta} = l$$

$$\frac{d}{dt}(r^2\dot{\theta}) = 0$$

dA' = infinitesimal area in XY plane

$$= r\,dr\,d\theta$$

$$\therefore \int_0^r dA' = \frac{r^2}{2}\,d\theta = dA$$

dA = area swept out by the particle

$$\Rightarrow \dot{A} = \frac{d}{dt}\left(\frac{r^2\theta}{2}\right)$$

So $\vec{L}$ is conserved is the same as saying

$$\frac{dA}{dt} = 0 \Rightarrow \int_{t_0}^{t+t_0} dA = \text{constant for any}$$

given value of t. This is Kepler's 2^{ND} law. True $\forall$ central forces

i.e. $V = V(r)$.

$$-\frac{\partial L}{\partial t} = \frac{dH}{dt} = 0 \Rightarrow H = E = \text{constant}$$

$$\Rightarrow \quad \frac{m\dot{\vec{r}}^2}{2} + V(r) = E$$

$$\Rightarrow \quad \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E$$

$$\dot{\theta} = l/(mr^2) \Rightarrow \frac{mr^2\dot{\theta}^2}{2} = \frac{l^2}{2mr^2}$$

$$\Rightarrow \quad \dot{r} = \frac{dr}{dt} = \left[\frac{2}{m}\left(E - V(r) - \frac{l^2}{2mr^2}\right)\right]^{1/2}$$

$$\Rightarrow \quad t = \int_{r_0}^{r} dr \left[\frac{2}{m}\left(E - V(r) - \frac{l^2}{2mr^2}\right)\right]^{-1/2}$$

$\Rightarrow$ we have $t = t(r)$ ~~are~~ or

inverting $r = r(t)$

Now $mr^2\dot{\theta} = l$ $\boxed{3\text{-}4.5}$

$$\Rightarrow \frac{d\theta}{dt} = \frac{l}{mr^2(t)}$$

$$\theta = \theta_0 + \frac{l}{m}\int_0^t \frac{dt}{r^2(t)}$$

We now have $r = r(t)$ & $\theta = \theta(t)$. Problem is solved. The integration constants are θ_0, r_0, E & l instead of the usual $\theta_0, r_0, \dot{\theta}_0, \dot{r}_0$. These 2 sets are equivalent.

Let $V_e \equiv V_{eff.} \equiv V(r) + \frac{l^2}{2mr^2}$

Consider a case where $V(r) = -k/r$

$$\therefore V_e = \frac{-k}{r} + \frac{l^2}{2mr^2}$$

$$\therefore \dot{r} = \left(\frac{2}{m}\right)^{1/2} \sqrt{E + \frac{k}{r} - \frac{l^2}{2mr^2}}$$

Now consider Figs 3.3 – 3.11 of the text. Physically meaningful solutions are those for which

$$\dot{r} \geq 0.$$

$$\Rightarrow \quad E + \frac{k}{r} - \frac{l^2}{2mr^2} = E - V_e(r) \geq 0$$

These are discussed in the Figs. listed above. Qualitatively the analysis in the Figs. holds whenever two conditions are satisfied: (i) $\frac{1}{r^2 V(r)} \rightarrow 0$ as $r \rightarrow \infty$

and (ii) $r^2 V(r) \rightarrow 0$ as $r \rightarrow 0$.

These conditions are satisfied for

$V(r) = -k/r$ but not satisfied for $V(r) = -k/r^4$ as shown in

Fig 3.9. In that case there exists a finite region $r_1 < r < r_2$ which is forbidden for ~~certa~~ certain energy values.

The Virial Theorem :$\longrightarrow$

Let $G = \sum_i \bar{p}_i \cdot \bar{r}_i$

$$\Rightarrow \dot{G} = \sum_i \left[\dot{\bar{p}}_i \cdot \bar{r}_i + \bar{p}_i \cdot \dot{\bar{r}}_i\right]$$

$$= \sum_i \left[\bar{F}_i \cdot \bar{r}_i + \frac{\bar{p}_i^{\,2}}{m_i}\right] = \sum_i \bar{F}_i \cdot \bar{r}_i + 2T$$

where T = kinetic energy.

$$I = \frac{1}{\tau}\int_0^\tau \dot{G}\,dt = \frac{1}{\tau}\left[G(\tau) - G(0)\right]$$

$$= \frac{2}{\tau}\int_0^\tau T\,dt + \frac{1}{\tau}\int_0^\tau \left(\sum_i \bar{F}_i \cdot \bar{r}_i\right) dt$$

$$= 2\langle T\rangle + \langle \sum_i \bar{F}_i \cdot \bar{r}_i \rangle$$

If the motion is periodic or is bounded

$$\Rightarrow \frac{1}{\tau}\left(G(\tau) - G(0)\right) = 0 \quad \text{as } \tau \to \infty$$

because G is bounded.

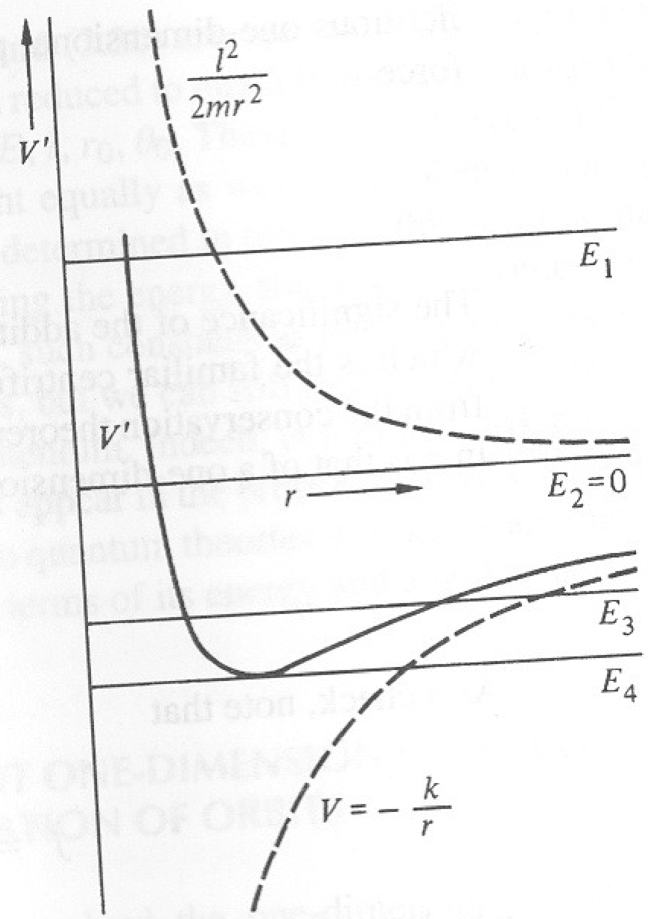


FIGURE 3.3 The equivalent one-dimensional potential for attractive inverse-square law of force.

the motion of a particle having the energy E_1, as shown in

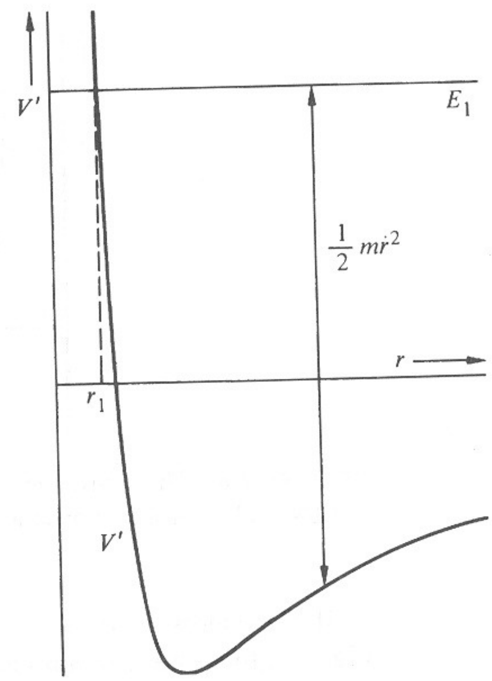


FIGURE 3.4 Unbounded motion at positive energies for inverse-square law of force.

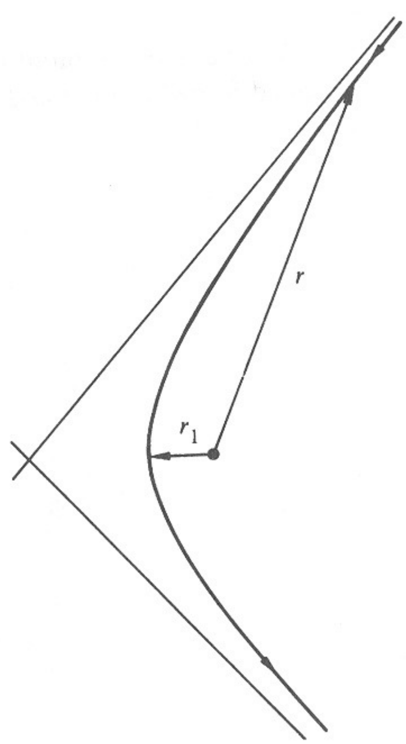


FIGURE 3.5 The orbit for E_1 corresponding to unbounded motion.

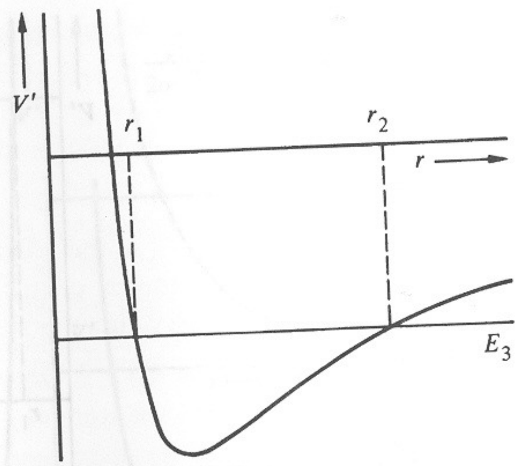


FIGURE 3.6 The equivalent one-dimensional potential for inverse-square law of force, illustrating bounded motion at negative energies.

If the energy is E_4 at the minimum of the fictitious potential as shown in Fig. 3.8, then the two bounds coincide. In such case, motion is possible at only one radius; $\dot{r} = 0$, and the orbit is a circle. Remembering that the effective "force" is the negative of the slope of the V' curve, the requirement for circular orbits is simply that f' be zero, or

$$f(r) = -\frac{l^2}{mr^3} = -mr\dot{\theta}^2.$$

We have here the familiar elementary condition for a circular orbit, that the applied force be equal and opposite to the "reversed effective force" of centripetal

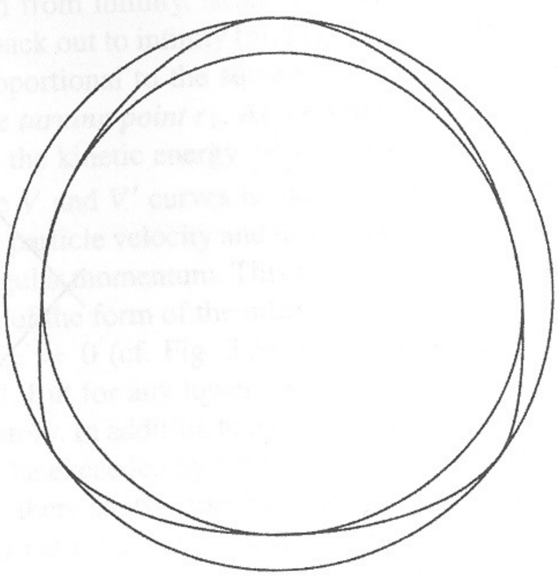


FIGURE 3.7 The nature of the orbits for bounded motion. ($\beta = 3$ from Section 3.6.)

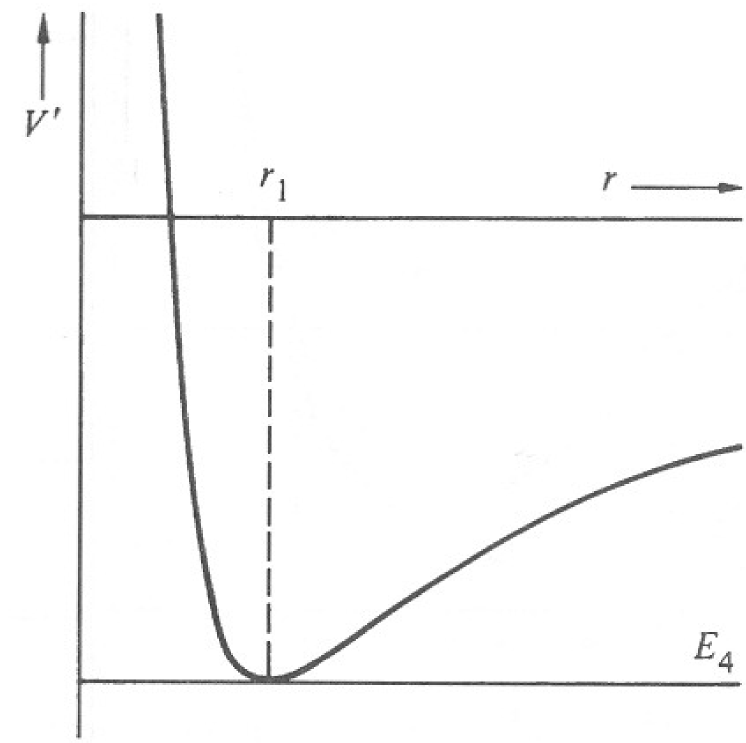


FIGURE 3.8 The equivalent one-dimensional potential of inverse-square law of force, illustrating the condition for circular orbits.

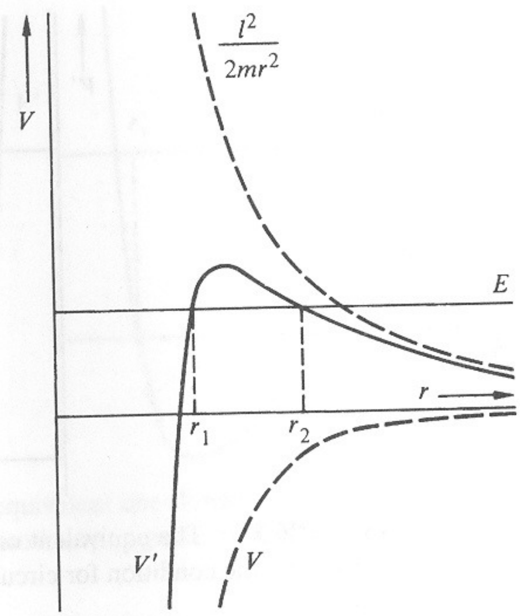


FIGURE 3.9 The equivalent one-dimensional potential for an attractive inverse-fou
law of force.

always remain so; the motion is unbounded, and the particle can never get insi
the "potential" hole. The initial condition $r_1 < r_0 < r_2$ is again not physica
possible.

Another interesting example of the method occurs for a linear restoring fo
(isotropic harmonic oscillator):

$$f = -kr, \qquad V = \tfrac{1}{2}kr^2.$$

For zero angular momentum, corresponding to motion along a straight line, V'
V and the situation is as shown in Fig. 3.10. For any positive energy the motio
bounded and, as we know, simple harmonic. If $l \neq 0$, we have the state of affa
shown in Fig. 3.11. The motion then is always bounded for all physically possi

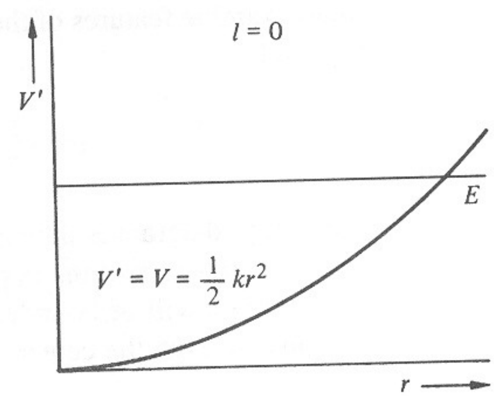


FIGURE 3.10 Effective potential for zero angular momentum.

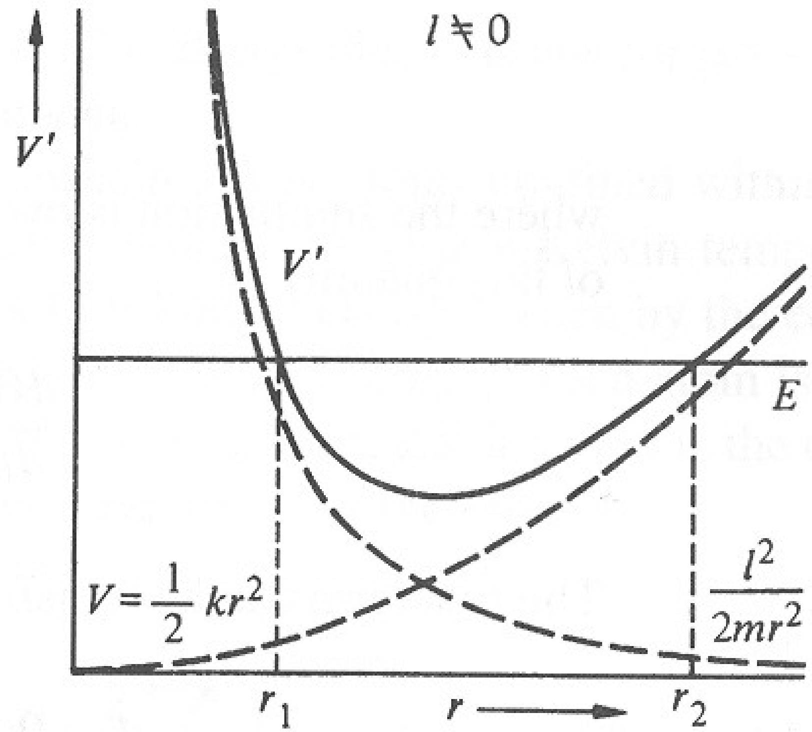


FIGURE 3.11 The equivalent one-dimensional potential for a linear restoring force.