Poisson Brackets

Let $F = F(\{q_i\},\{p_i\},t)$

Then
$$\dot{F} \equiv \frac{dF}{dt} = \frac{\partial F}{\partial t} + \sum_i \left[\frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial p_i}\dot{p}_i\right]$$

$$= \frac{\partial F}{\partial t} + \sum_i \left[\frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i}\right]$$

$$= \frac{\partial F}{\partial t} + [F,H]_{q,p}$$

where we define a Poisson Bracket (PB) as

$$[A,B]_{q,p} \equiv \sum_i \left(\frac{\partial A}{\partial q_i}\right)\left(\frac{\partial B}{\partial p_i}\right) - \left(\frac{\partial B}{\partial q_i}\right)\left(\frac{\partial A}{\partial p_i}\right)$$

We can show several properties immediately

$$[q_i,q_j]_{q,p} = \sum_k \left[\frac{\partial q_i}{\partial q_k}\frac{\partial q_j}{\partial p_k} - \frac{\partial q_j}{\partial q_k}\frac{\partial q_i}{\partial p_k}\right]$$

$$= 0$$

Similarly $[p_i,p_j]_{q,p} = 0$

and
$$[q_i,p_j]_{q,p} = \sum_k \left[\frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial p_j}{\partial q_k}\frac{\partial q_i}{\partial p_k}\right]$$

$$= \sum_k \left[\delta_{ik}\delta_{jk} - 0\right] = \delta_{ij}$$

Thus $[q_i, q_j]_{q,p} = [p_i, p_j]_{q,p} = 0$

and $[q_i, p_j]_{q,p} = \delta_{ij}$

are called the Fundamenta Poisson Brackets (FPB). Unless specified otherwise we will assume PBs to be calculated w.r.to $\{q_i\}$ and $\{p_i\}$

i.e. $[A, B] \equiv [A, B]_{q,p}$

The following may also be shown easily

$$[A, B] + [B, A] = 0 \quad \Big\} \text{antisymmetry}$$

$$[C_1 f + C_2 g, h] = C_1[f, h] + C_2[g, h] \quad \Big\} \text{linearity}$$

where C_1 & C_2 are constants

$$[f, gh] = [f, g]h + [f, h]g$$

Tedious ~~algra~~ algebra also leads to ~~Ja~~ Jacobi's identity

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0$$

or $[[g, h], f] + [[h, f], g] + [[f, g], h] = 0$

We now evaluate

$$\frac{d}{dt}[f,g] = \big[[f,g],H\big] + \frac{\partial}{\partial t}[f,g]$$

It is easily seen that

$$\frac{\partial}{\partial t}[f,g] = \left[\frac{\partial f}{\partial t}, g\right] + \left[f, \frac{\partial g}{\partial t}\right]$$

By Jacobi's identity we get

$$\big[[f,g],H\big] = -\big[[g,H],f\big] - \big[[H,f],g\big]$$

$$= \big[f,[g,H]\big] + \big[[f,H],g\big]$$

$$\Rightarrow \frac{d}{dt}[f,g] = \big[[f,H],g\big] + \left[\frac{\partial f}{\partial t}, g\right]$$

$$+ \big[f,[g,H]\big] + \left[f, \frac{\partial g}{\partial t}\right]$$

$$= \left[[f,H] + \frac{\partial f}{\partial t}, g\right] + \left[f, [g,H] + \frac{\partial g}{\partial t}\right]$$

$$\frac{d}{dt}[f,g] = \left[\frac{df}{dt}, g\right] + \left[f, \frac{dg}{dt}\right]$$

If f and g are constants

$\Rightarrow [f,g]$ is also constant.

$\therefore$ if f and g are constants of motion

i.e. $\dot{f} = \dot{g} = 0$ then

$$\frac{d}{dt}[f, g] = 0.$$

Canonical Transformations :$\rightarrow$

Consider a transformation

$$Q_i = Q_i(\{q_i\}, \{p_i\}, t) \quad , \forall i$$

$$P_i = P_i(\{q_i\}, \{p_i\}, t) \quad , \forall i$$

The initial Hamiltonian of the system is

$$H = H(\{q_i\}, \{p_i\}, t)$$

The new Hamiltonian is

$$K = K(\{Q_i\}, \{P_i\}, t)$$

The initial equations of motion are

$$\dot{q}_i = +\frac{\partial H}{\partial p_i} \quad , \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

If with the new co-ordinates we get

the same form of the equations

i.e. $\dot{Q}_i = \frac{\partial K}{\partial P_i}$, $\dot{P}_i = \frac{-\partial K}{\partial Q_i}$

then the transformation is called a canonical transformation (CT).

Note in chapter 8 we defined

$\bar{\eta} \ni \eta_i = q_i , \eta_{i+n} = p_i , \forall i = 1, \dots n.$

$\bar{\eta}$ is a $(2n \times 1)$ column matrix

$\frac{\partial H}{\partial \bar{\eta}}$ is a $(2n \times 1)$ column matrix

$$\ni \left(\frac{\partial H}{\partial \bar{\eta}}\right)_i = \frac{\partial H}{\partial q_i} , \left(\frac{\partial H}{\partial \bar{\eta}}\right)_{i+n} = \left(\frac{\partial H}{\partial p_i}\right)$$

$\forall i = 1, \dots n.$

$\bar{\bar{J}}$ has been defined by

$$\bar{\bar{J}} \equiv \begin{bmatrix} \bar{\bar{O}} & \bar{\bar{I}} \\ -\bar{\bar{I}} & \bar{\bar{O}} \end{bmatrix} \quad \text{where } \bar{\bar{I}}_{ij} = \delta_{ij}$$

$\bar{O}_{ij} = 0 \quad , \forall i, j$

$$\therefore \bar{\bar{J}}_{ij} \equiv \delta_{j,i+n} - \delta_{j,i-n} \quad , \forall i = 1, \dots 2n$$
$$j = 1, \dots 2n.$$

In this form Hamiltons Equations are

$$\dot{\bar{\eta}} = \bar{\bar{J}} \cdot \left(\frac{\partial H}{\partial \bar{\eta}}\right)$$

A restricted canonical transformation (RCT) is one in which time does not appear explicitly

i.e. $Q_i = Q_i(\{q_i\},\{p_i\})$

$P_i = P_i(\{q_i\},\{p_i\})$

Let us define $\bar{J}$ as a $(2n \times 1)$ column,

$\Rightarrow$ $\bar{J}_i = Q_i$ and $\bar{J}_{i+n} = P_i$, $\forall i = 1, 2, \ldots n$.

$\Rightarrow$ $\bar{J} = \bar{J}(\bar{\eta})$ for a RCT.

$$\dot{\bar{J}}_i = \sum_j \frac{\partial \bar{J}_i}{\partial \bar{\eta}_j} \dot{\bar{\eta}}_j$$

Let $\bar{\bar{M}}_{ij} \equiv \dfrac{\partial \bar{J}_i}{\partial \bar{\eta}_j}$ $\Rightarrow$ $\dot{\bar{J}} = \bar{\bar{M}} \dot{\bar{\eta}}$

$$\therefore \quad \dot{\bar{J}} = \bar{\bar{M}} \cdot \bar{\bar{J}} \cdot \left(\frac{\partial H}{\partial \bar{\eta}}\right)$$

Now $\dfrac{\partial H}{\partial \bar{\eta}_i} = \sum_j \dfrac{\partial H}{\partial \bar{\zeta}_j}\left(\dfrac{\partial \bar{\zeta}_j}{\partial \eta_i}\right)$

$$= \sum_j \frac{\partial H}{\partial \bar{\zeta}_j}\,\bar{\bar{M}}_{ji} = \sum_j \left(\bar{\bar{M}}^T\right)_{ij}\left(\frac{\partial H}{\partial \bar{\zeta}_j}\right)$$

$$\therefore \quad \frac{\partial H}{\partial \bar{\eta}} = \boxed{\cancel{\bar{\bar{M}}^T\left(\frac{\partial H}{\partial \bar{\eta}}\right)}} \quad \bar{\bar{M}}^T\left(\frac{\partial H}{\partial \bar{\zeta}}\right)$$

$$\therefore \quad \dot{\bar{\zeta}} = \bar{\bar{M}}\,\bar{\bar{J}}\,\bar{\bar{M}}^T\left(\frac{\partial H}{\partial \bar{\zeta}}\right)$$

We need $\dot{\bar{\zeta}} = \bar{\bar{J}}\left(\dfrac{\partial H}{\partial \bar{\zeta}}\right)$ for a canonical transformation

$$\ni \quad \bar{\bar{M}}\,\bar{\bar{J}}\,\bar{\bar{M}}^T = \bar{\bar{J}} \longrightarrow (9.55)$$

Eq. (9.55) is a necessary and sufficient condition for a transformation to be canonical. Note that there is a trivial scale transformation $Q_i = \lambda q_i$ and $P_i = \lambda p_i$ which is canonical but we ignore in Eq. (9.55). Otherwise (9.55) modifies to $\bar{\bar{M}}\,\bar{\bar{J}}\,\bar{\bar{M}}^T = \lambda \bar{\bar{J}}$. We will always take $\lambda = 1$.

Eq. (9.55) is called the symplectic condition.

This condition says that for a transformation to be (CT) we need FPB to be invariant

i.e. $$[\varphi_i, \varphi_j]_{q,p} = [P_i, P_j]_{q,p} = 0$$

and $$[\varphi_i, P_j]_{q,p} = \delta_{ij}$$

This proves that a canonical transformation leaves FPB invariant.
Now we can prove that any arbitrary PB is invariant under a (CT)
We want to prove

$$[f, g]_{q,p} = [f, g]_{\varphi, P}$$

$$[f,g]_{q,p} = \sum_j \left[\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j}\right]$$

$$= \sum_j \sum_k \left\{ \frac{\partial f}{\partial q_j}\left[\frac{\partial g}{\partial \varphi_k}\frac{\partial \varphi_k}{\partial p_j} + \frac{\partial g}{\partial P_k}\frac{\partial P_k}{\partial p_j}\right]\right.$$

$$\left. - \frac{\partial f}{\partial p_j}\left[\frac{\partial g}{\partial \varphi_k}\frac{\partial \varphi_k}{\partial q_j} + \frac{\partial g}{\partial P_k}\frac{\partial P_k}{\partial q_j}\right]\right.$$

$$= \sum_k \left[\frac{\partial g}{\partial \varphi_k}[f, \varphi_k]_{q,p} + \frac{\partial g}{\partial P_k}[f, P_k]_{q,p}\right]$$

$$\therefore [f,g]_{q,p} = \sum_k \left[\frac{\partial g}{\partial \varphi_k} [F,\varphi_k]_{q,p} + \frac{\partial g}{\partial P_k} [f,P_k]_{q,p} \right]$$

If we choose $f = \varphi_k$ in the above we get

$$[\varphi_k, g]_{q,p} = \left(\frac{\partial g}{\partial P_k} \right) = -[g, \varphi_k]$$ $\boxed{9\text{-}9}$

If we choose $f = P_k$ we will get

$$[P_k, g] = \left(-\frac{\partial g}{\partial \varphi_k} \right) = -[g, P_k]$$

If we flip f and g we get with the above results

$$[g,f]_{q,p} = \sum_k \left[\frac{\partial f}{\partial \varphi_k} [g, \varphi_k]_{q,p} + \frac{\partial f}{\partial P_k} [g, P_k]_{q,p} \right]$$

$$= \sum_k \left[\left(-\frac{\partial f}{\partial \varphi_k} \frac{\partial g}{\partial P_k} \right) + \frac{\partial f}{\partial P_k} \frac{\partial g}{\partial \varphi_k} \right]$$

$$= [g,f]_{\varphi,P}$$

$$\Rightarrow [g,f]_{q,p} = [g,f]_{\varphi,P} \quad \text{or}$$

$$[f,g]_{q,p} = [f,g]_{\varphi,P} \quad \text{as required.}$$

Generating functions :→

Consider Hamiltons principle

$$\delta \int_{t_1}^{t_2} L\, dt = 0$$

written either as

$$\delta \int_{t_1}^{t_2} \left[\sum_i p_i \dot{q}_i - H(\{q_i\},\{p_i\},t) \right] dt = 0$$

and $$\delta \int_{t_1}^{t_2} \left[\sum_i P_i \dot{Q}_i - K(\{Q_i\},\{P_i\},t) \right] dt = 0$$

where Q_i and P_i and K are related

by a CT to q_i, p_i and H.

Subtracting the two we get

$$\delta \int_{t_1}^{t_2} \left[\sum_i p_i \dot{q}_i - H - \sum_i P_i \dot{Q}_i + K \right] dt = 0$$

We can identify a generating function

$F \ni$

$$\frac{dF}{dt} = \sum_i \left(p_i \dot{q}_i - P_i \dot{Q}_i \right) + K - H$$

so that we get

$$\delta \int_{t_1}^{t_2} \frac{dF}{dt}\, dt = \delta\left[F(t_2) - F(t_1)\right]$$

But at t_2 and t_1 all variations in q_i, p_i, φ_i and P_i are chosen to be zero.

Hence $\delta \cancel{\delta}\left[F(t_2) - F(t_1)\right] = 0$.

F is in general a ~~func~~ function of all q_i, p_i, φ_i, P_i, $4n$ in number only $2n$ of which are ~~indep~~ independent. It is also a function of t in general.

There are only 4 possibilities for the functional forms of F: $F_1(\{q_i\}, \{\varphi_i\}, t)$

$F_2(\{q_i\}, \{P_i\}, t)$, $F_3(\{p_i\}, \{\varphi_i\}, t)$ and

$F_4(\{p_i\}, \{P_i\}, t)$ related to each other

by Legendre Transformations or (LTs).

Consider $F = F_1(\{q_i\}, \{\varphi_i\}, t)$

$$\therefore \frac{dF}{dt} = \sum_i \left[\frac{\partial F_1}{\partial q_i}\dot{q}_i + \frac{\partial F_1}{\partial \varphi_i}\dot{\varphi}_i\right] + \frac{\partial F_1}{\partial t}$$

$$= \sum_i \left(p_i \dot{q}_i + P_i \dot{\varphi}_i\right) + K - H$$

$$\Rightarrow \frac{\partial F_1}{\partial t} \equiv K - H$$

$$p_i = \frac{\partial F_1}{\partial q_i} \quad \text{and} \quad P_i = \frac{\partial F_1}{\partial \varphi_i}$$

Doing an LT on F_1 we get

$F_2 = F_1 + \sum_i P_i \varphi_i$ so that

$F_2 = F_2(\{q_i\}, \{P_i\}, t)$

A different LT on F_1 gives

$F_3 = F_1 - \sum_j p_j q_j \ni F_3 = F_3(\{p_i\}, \{\varphi_i\}, t)$

Two LTs on F_1 give

$F_4 = F_1 + \sum_i (P_i \varphi_i - q_i p_i) \ni F_4 = F_4(\{p_i\}, \{P_i\}, t)$

These (LT)s are not always possible as we will see in an example (if time permits!) Also a function could be of type F_1 for ~~so~~ the i^{TH} degree of freedom and type F_2 in the j^{TH} degree of freedom.

Coming up with appropriate F functions requires intuition and there is not a set recipe.