We will now call $\lambda_k \equiv I_k$, $k=1,2,3$.

A rigid body with $I_1 = I_2 = I_3$ is called a spherical top.

If $I_1 = I_2 \neq I_3$ then its a symmetrical top and $I_1 \neq I_2 \neq I_3 \neq I_1$ corresponds to an asymmetrical top. If $I_3 = 0$ and $I_1 = I_2$ it is called a rotor. 5-15

The Euler Equations of Motion :→

For holonomic conservative systems the Lagrangian for a rigid body may be broken into two independent parts

$$L(\{q_i\},\{\dot{q}_i\}) = L_c(\{q_c\},\{\dot{q}_c\}) + L_b(\{q_b\},\{\dot{q}_b\})$$

$c \equiv$ center of mass (COM),
$b \equiv$ body with respect to COM.

So we may only consider $L_b(\{q_b\},\{\dot{q}_b\})$.

Let the origin of the fixed space axes be at the fixed point on the rigid body or at its COM. In both cases we get

$$\left(\frac{d\bar{L}}{dt}\right)_s = \bar{N} = \text{the torque}.$$

But $\left(\frac{d\bar{L}}{dt}\right)_s = \left(\frac{d\bar{L}}{dt}\right)_b + \bar{\omega} \times \bar{L}$

$$\Rightarrow \left(\frac{d\bar{L}}{dt}\right) + \bar{\omega} \times \bar{L} = \bar{N} \quad \rightarrow (5.37)$$

where we have dropped the subscript b. We will work in the body fixed axes henceforth. ~~If we~~

5-16

If we now choose the body axes as the principal axes then

$$\bar{L} = \bar{\bar{I}}\,\bar{\omega} \Rightarrow L_i = I_i \omega_i$$

Eq. (5.37) then becomes

$$I_i \dot{\omega}_i + \sum_k \epsilon_{ijk}\, \omega_j L_k = N_i$$

$$\Rightarrow I_i \dot{\omega}_i + \sum_k \epsilon_{ijk}\, \omega_j \omega_k I_k = N_i \quad \rightarrow (5.39)$$

$$\Rightarrow \left.\begin{aligned} I_1 \dot{\omega}_1 - \omega_2 \omega_3 (I_2 - I_3) &= N_1 \\ I_2 \dot{\omega}_2 - \omega_3 \omega_1 (I_3 - I_1) &= N_2 \\ I_3 \dot{\omega}_3 - \omega_1 \omega_2 (I_1 - I_2) &= N_3 \end{aligned}\right\} (5.39')$$

If there is no torque on the system then $\bar{N} = \bar{0}$.

Case (i), $\dot{\bar{\omega}} = \bar{0}$, $\bar{N} = \bar{0}$

$\Rightarrow$ if $I_1 \neq I_2 \neq I_3 \neq I_1$ then we need at least two of $\omega_1, \omega_2, \omega_3$ to be zero
Choose $\omega_1 = \omega_2 = 0$

$\boxed{5\text{-}17}$

$$\Rightarrow I_3 \dot{\omega}_3 = 0 \Rightarrow \omega_3 = \text{constant}.$$

This case shows that for an asymmetric body $\dot{\bar{\omega}} = \bar{0} \Leftrightarrow$ rotation is about one ~~on~~ of the principal ~~ea~~ axes only.

Case (ii), $I_1 = I_2 \neq I_3$, $\bar{N} = \bar{0}$

$$\Rightarrow I_3 \dot{\omega}_3 = 0 \Rightarrow \omega_3 = \text{constant}$$

$$\therefore \dot{\omega}_1 = -\Omega \omega_2, \quad \dot{\omega}_2 = \Omega \omega_1 \longrightarrow (5.48)$$

where $\Omega \equiv \left(\dfrac{I_3 - I_1}{I_1} \right) \omega_3 \longrightarrow (5.49)$

Eq. (5.48) gives $\omega_2 = \dfrac{-\dot{\omega}_1}{\Omega}$ and $\ddot{\omega}_1 = -\Omega^2 \omega_1$

$$\Rightarrow \omega_1 = A\cos(\Omega t + \phi), \quad \omega_2 = A\sin(\Omega t + \phi).$$

$$\Rightarrow |\bar{\omega}| = \sqrt{A^2 + \omega_3^2} = \text{constant}$$

$$L = \sqrt{I_1^2 A^2 + I_3^2 \omega_3^2} = \text{constant}$$

$$T = \frac{1}{2}\left[I_1 A^2 + I_3 \omega_3^2 \right] = \text{constant}$$

$\bar{\omega} - \omega_3\hat{k} = \omega_1\hat{i} + \omega_2\hat{j}$ precesses around the z axis of the body principal axes.

This is shown in Fig 5.6. $\boxed{5\text{-}18}$

Such a procession occurs for the earth for which $\dfrac{I_3 - I_1}{I_1} = 3.27 \times 10^{-3}$

$$\therefore \Omega = \left(\frac{I_3 - I_1}{I_1}\right)\omega_3 = \frac{3.27 \times 10^{-3}}{24\text{ hrs}} \times 2\pi .$$

Here we assume $\omega_3 = 2\pi/(24\text{hrs})$ and approximate the earth's daily rotation as being ~~force~~ torque free. The situation is more complicated due to subtleties in considering the symmetries of mass distribution.

Motion of a top with one point fixed

Fig 5.7 shows such a top.

The kinetic energy of such a top is

$$T = \frac{1}{2} I_1 (\omega_1^2 + \omega_2^2) + \frac{I_3}{2}\omega_3^2$$

where by symmetry $I_1 = I_2$.

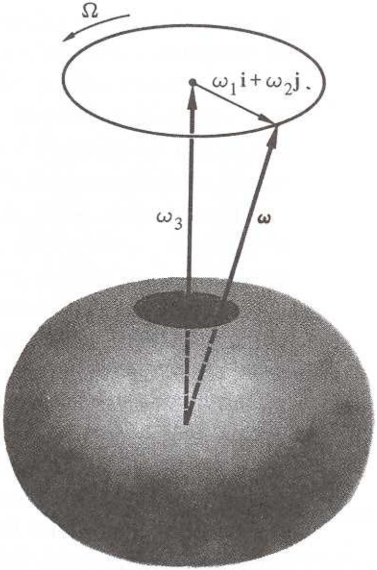

FIGURE 5.6 Precession of the angular velocity about the axis of symmetry in the force-free motion of a symmetrical rigid body.

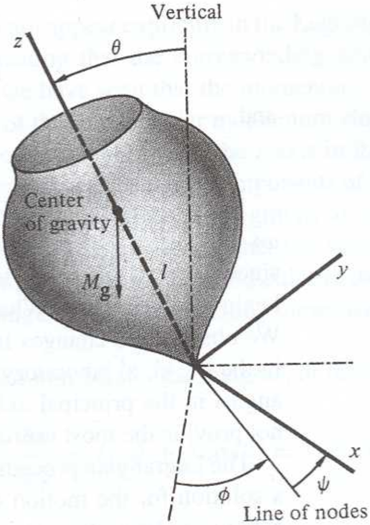

FIGURE 5.7 Euler's angles specifying the orientation of a symmetrical top.

From Eqs. (4.87) we get

$$\omega_1 = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi$$

$$\omega_2 = \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi \qquad \boxed{5\text{-}21}$$

$$\omega_3 = \dot{\phi} \cos\theta + \dot{\psi}$$

$$\therefore T = \frac{I_1}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + \frac{I_3}{2}\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 \rightarrow (5.50)$$

The potential energy

$$V = -\sum_i m_i \bar{r}_i \cdot \bar{g} = -M\bar{R}\cdot\bar{g} = Mgl\cos\theta$$

M = mass of the top.

$$\therefore L = \frac{I_1}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + \frac{I_3}{2}\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - Mgl\cos\theta$$

$$\frac{\partial L}{\partial \psi} = \frac{\partial L}{\partial \phi} = 0$$

$$\Rightarrow p_\psi \equiv \frac{\partial L}{\partial \dot{\psi}} = I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right) = I_1 a \rightarrow (5.53)$$

where a = constant

$$p_\phi \equiv \frac{\partial L}{\partial \dot{\phi}} = \left(I_1 \sin^2\theta + I_3 \cos^2\theta\right)\dot{\phi} + I_3 \dot{\psi} \cos\theta = I_1 b \rightarrow (5.54)$$

where

b = another constant.

Eliminating $\dot{\psi}$ from Eqs. (5.53) and (5.54) gives

$$\dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta} \longrightarrow (5.57)$$

Eliminating $\dot{\phi}$ from Eqs. (5.57) and (5.54) & (5.53) gives

$$\dot{\psi} = \frac{I_1 a}{I_3} - \left(\frac{b - a\cos\theta}{\sin^2\theta}\right)\cos\theta \longrightarrow (5.58)$$

The system is conservative $\Rightarrow T+V=E$

$$\therefore E = \frac{I_1}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + \frac{I_3 \omega_3^2}{2} + Mgl\cos\theta$$

$E =$ constant, from Eq. (5.53) we see that $\omega_3 =$ constant,
From Eq. (5.57) we get

$$E' \equiv E - \frac{1}{2} I_3 \omega_3^2 = \frac{I_1}{2}\dot{\theta}^2 + V_e(\theta)$$

where $V_e(\theta) \equiv Mgl\cos\theta + \dfrac{(b - a\cos\theta)^2}{2\sin^2\theta}$

$$\therefore \dot{\theta} = \sqrt{\frac{2}{I_1}\left(E' - V_e(\theta)\right)}$$

$$\Rightarrow t(\theta) = \left(\frac{I_1}{2}\right)^{1/2} \int \frac{d\theta}{\sqrt{E' - V(\theta)}} \longrightarrow (5.63)$$

Inverting Eq. 5.63 we can get

$\theta = \theta(t)$ and then using this in Eqs. 5.57 and 5.58 we can get

$$\phi(t) = \int \frac{b - a\cos(\theta(t))}{\sin^2(\theta(t))}$$

and $$\psi(t) = \frac{I_1 a t}{I_3} - \int \frac{[b - a\cos(\theta(t))]\cos(\theta(t))}{\sin^2(\theta(t))}$$

The effective potential $V_e(\theta)$ is as shown in the Fig.

Case (1) $E' = E'_2 \Rightarrow \left.\frac{dV_e}{d\theta}\right|_{\theta=\theta_0} = 0$, $V_e(\theta_0) = \text{minimum}$

$$\left.\frac{dV_e}{d\theta}\right|_{\theta=\theta_0} = -Mgl\sin\theta_0 + \frac{4\sin^2\theta(b-a\cos\theta)(a\sin\theta_0)}{4\sin^4\theta_0}$$

$$-\frac{4(b-a\cos\theta_0)^2\sin\theta_0\cos\theta_0}{4\sin^4\theta_0} = 0$$

Let $\beta = (b - a\cos\theta)$

$$\Rightarrow (\cos\theta_0)\beta^2 - (a\sin^2\theta_0)\beta + Mgl\sin^4\theta_0 = 0$$

$$\Rightarrow \beta_\pm = \frac{a\sin^2\theta_0}{2\cos\theta_0}\left[1 \pm \sqrt{1 - \frac{4Mgl\cos\theta_0}{a^2}}\right]$$

$\beta_\pm \in R$

$\Rightarrow \quad a^2 \geq 4Mgl\cos\theta_0$

$\Rightarrow$ with (5.53) that

$$\frac{I_3^2\omega_3^2}{I_1^2} \geq 4Mgl\cos\theta_0$$

$$\Rightarrow \quad \omega_3 \geq \frac{2I_1}{I_3}\left[\sqrt{Mgl\cos\theta_0}\right]$$

$\Rightarrow$ for a constant θ we need a minimum value of ω_3.

$$\dot{\phi}_\pm = \frac{\beta_\pm}{\sin^2\theta_0} \quad \text{and}$$

$$\dot{\psi}_\pm = \frac{I_1 a}{I_3} - \frac{\beta_\pm\cos\theta_0}{\sin^2\theta_0}$$

Case (ii) $E' = E'_1 \Rightarrow \theta_1 \leq \theta \leq \theta_2$.

$\Rightarrow$ θ changes with time between the two limits. This is called nutation. This is shown in Fig 5.9, by the path described by the projection of the body symmetry axis on a unit sphere in the fixed system.

When $\dot{\phi} > 0$ or $\dot{\phi} < 0$ i.e. $\dot{\phi}$ never changes sign the Fig 5.9a is applicable to the motion. If $\dot{\phi}$ changes sign once between θ_1 and θ_2 then Fig 5-9b is applicable. If $\dot{\phi}\big|_{\theta=\theta_1} = 0$

$$\Rightarrow \quad b - a\cos\theta_1 = 0$$

$$\Rightarrow \quad \dot{\phi}\big|_{\theta=\theta_1} = \dot{\theta}\big|_{\theta=\theta_1} = 0.$$

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These are the initial conditions along with $\theta = \theta_1$ for starting a toy top usually. Then Fig 5-9c is applicable.

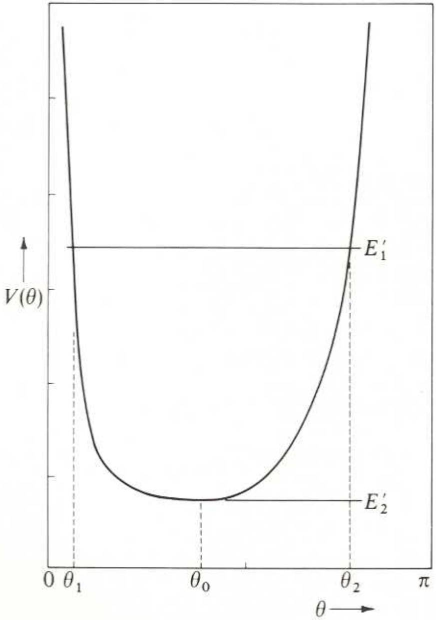

Figure 10-14

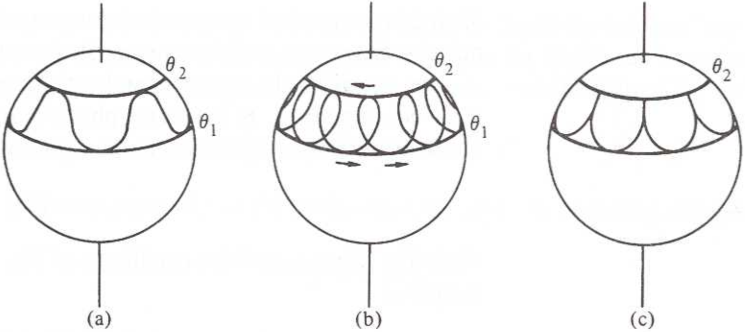

FIGURE 5.9 The possible shapes for the locus of the figure axis on the unit sphere.