Kinematics of Rigid Bodies

A rigid body is a system of mass points subject to the constraints (which are holonomic) that the distances between all pairs of points ~~are~~ remain constant. $\boxed{4\text{-}1}$

Kinematics is the study of motion of particles without reference to mass or force i.e. to the causes of motion.

If $\exists$ N particles then we get

$$\bar{r}_{ij} \equiv \bar{r}_i - \bar{r}_j$$

$$\text{and} \quad \sqrt{\bar{r}_{ij}^{\,2}} = c_{ij} \quad , \forall i,j \in 1,2,\ldots N, \quad (4.1)$$

$c_{ij} =$ constant $\forall i,j$.

$\Rightarrow$ for 3N degrees of freedom $\exists \frac{N}{2}(N-1)$ constraints. Usually $3N < \frac{N}{2}(N-1)$

$\Rightarrow$ some constraints are not independent

$\Rightarrow$ $\exists$ only 6 independent co-ordinates

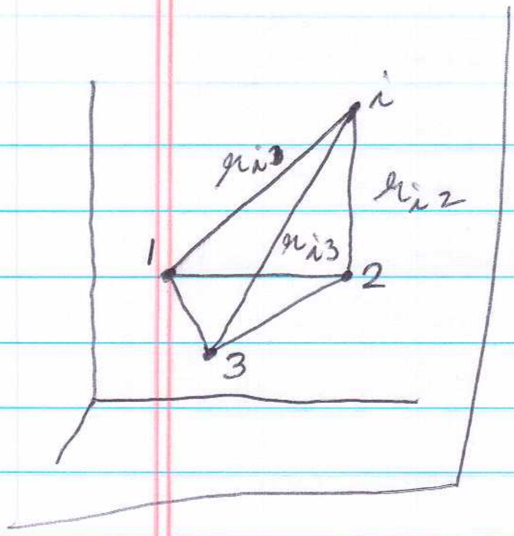

Fig 4.1 shows that if 3 non-collinear points are fixed then the i^{th} point is fixed by 3 constraints

$$\bar{r}_{i3}^{\,2} = c_{i3}^2 , \quad \bar{r}_{i2}^{\,2} = c_{i2}^2 , \quad \bar{r}_{i1}^{\,2} = c_{i1}^2$$

$\exists$ only 9 coordinates which are independent. However these are further reduced to 6 by the 3 constraints

$$\bar{r}_{ij}^{\,2} = c_{ij}(1-\delta_{ij}) \quad , \forall i,j \in 1,2,3 .$$

Consider XYZ frame as fixed in the laboratory. Let $X'Y'Z'$ be fixed on the rigid body. Let $\hat{i}, \hat{j}$ & $\hat{k}$ be unit vectors along $+(XYZ)$ respectively while $\hat{i}', \hat{j}', \hat{k}'$ be those along $+(X'Y'Z')$ respectively. Also let

$$(XYZ) \equiv (x_1 x_2 x_3) \quad , \quad (X'Y'Z') \equiv (x_1' x_2' x_3')$$
$$\hat{i} \equiv \hat{e}_1 , \ \hat{j} \equiv \hat{e}_2 , \ \hat{k} \equiv \hat{e}_3 \quad , \quad \hat{i}' \equiv \hat{e}_1' , \ \hat{j}' \equiv \hat{e}_2' , \ \hat{k}' \equiv \hat{e}_3'$$

Define angle θ_{ij} as the angle between $\hat{e}_i$ and $\hat{e}_j'$, $\forall i = 1,2,3,\ j = 1,2,3$

$$\Rightarrow \cos\theta_{ij} = \hat{e}_i \cdot \hat{e}_j'$$

$\therefore$ any vector $\overline{G}$ may be expressed as

$$\overline{G} = \sum_{i=1}^{3} G_i \hat{e}_i \quad \text{or} \quad \overline{G} = \sum_{i=1}^{3} G'_i \hat{e}'_i$$

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$$G'_i = \overline{G} \cdot \hat{e}'_i = \sum_{j=1}^{3} G'_j \left(\hat{e}_j \cdot \hat{e}'_i\right)$$

$$\text{or} \quad G'_i = \overline{G} \cdot \hat{e}'_i = \sum_{j=1}^{3} G_j \left(\hat{e}_j \cdot \hat{e}'_i\right)$$

$$= \sum_{j=1}^{3} G_j \cos(\theta_{ij}) \longrightarrow 4.6$$

$$\hat{e}_i \cdot \hat{e}_j = \hat{e}'_i \cdot \hat{e}'_j = \delta_{ij} \longrightarrow 4.7$$

Now $\overline{G} = \sum_j \left(\overline{G} \cdot \hat{e}_j\right) \hat{e}_j = \sum_j \left(\overline{G} \cdot \hat{e}'_j\right) \hat{e}'_j$

$$\therefore \quad \hat{e}'_m = \sum_l \left(\hat{e}'_m \cdot \hat{e}_l\right) \hat{e}_l$$

$$\Rightarrow \quad \hat{e}'_m \cdot \hat{e}'_n = \sum_l \left(\hat{e}'_m \cdot \hat{e}_l\right)\left(\hat{e}_l \cdot \hat{e}'_n\right)$$

$$= \sum_l \cos(\theta_{lm}) \cos(\theta_{ln}) = \hat{e}'_m \cdot \hat{e}'_n = \delta_{mn}$$

$$\Rightarrow \quad \sum_l \cos\theta_{lm} \cos\theta_{ln} = \delta_{mn} \longrightarrow 4.8 \text{ or } 4.9$$

Orthogonal Transformations : $\rightarrow$

A linear transformation from $\{x_i\}$ to $\{x'_i\}$ is one which has a form

$$x_i' = \sum_j A_{ij} x_j$$

In matrix form

$$\bar{x}' = \bar{\bar{A}} \bar{x}$$

In general for vectors $\bar{g}$ in the old (XYZ) frame we get $\bar{g}'$ in the (X'Y'Z') frame with the ~~sa~~ transformation

$$\bar{g}' = \bar{\bar{A}} \bar{g}$$

In matrix form we get for the length of vectors to be preserved under $\bar{\bar{A}}$ the condition

$$\bar{g}'^T \bar{g}' = \bar{g}^T \bar{g}$$

$$\Rightarrow \quad \bar{g}'^T \bar{g}' = \bar{g}^T \bar{\bar{A}}^T \bar{\bar{A}} \bar{g} = \bar{g} \cdot \bar{g}$$

$$\Rightarrow \quad \bar{\bar{A}}^T \bar{\bar{A}} = \bar{\bar{I}}$$

$$\Rightarrow \quad \sum_{j=1}^{3} (a^T)_{ij} \, a_{jk} = \delta_{ik}$$

$$\Rightarrow \quad \sum_{j=1}^{3} a_{ji} \, a_{jk} = \delta_{ik} \longrightarrow (4.15)$$

Such transformations are called orthogonal transformations.

Also $\quad \bar{\bar{A}}^T \bar{\bar{A}} = \bar{\bar{I}} \Rightarrow \bar{\bar{A}}^T (\bar{\bar{A}} \bar{\bar{A}}^{-1}) = \bar{\bar{I}} \bar{\bar{A}}^{-1}$

Consider a transformation that keeps x_3 unchanged

$$x_3' = x_3 \quad , \quad x_3' = \sum_{j=1} a_{3j} x_j$$

$$\Rightarrow \quad x_3' = a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = x_3$$

$$\Rightarrow \quad a_{31} = a_{32} = 0 \, , \quad a_{33} = 1$$

Also

$$\bar{x}' = \bar{\bar{A}} \bar{x}$$

$$\Rightarrow \quad \bar{\bar{A}}^{-1} \bar{x}' = \bar{\bar{A}}^{-1} \bar{\bar{A}} \bar{x} = \bar{x}$$

$$\Rightarrow \quad \bar{\bar{A}}^{T} \bar{x}' = \bar{x}$$

$$\Rightarrow \quad \bar{x} = \bar{\bar{A}}^{T} \bar{x}' = \bar{\bar{A}}^{-1} \bar{x}'$$

$$\Rightarrow \quad x_3 = \sum_{j=1}^{3} a_{j3} x_j' = a_{13} x_1' + a_{23} x_2' + a_{33} x_3'$$

$$\Rightarrow \quad a_{13} = a_{23} = 0$$

$$\Rightarrow \quad \bar{\bar{A}} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\bar{\bar{A}}^{T} = \bar{\bar{A}}^{-1} \Rightarrow \begin{bmatrix} a_{11} & a_{21} & 0 \\ a_{12} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad a_{11}^2 + a_{21}^2 = 1 = a_{22}^2 + a_{12}^2$$

and $\quad a_{11}a_{12} + a_{21}a_{22} = 0$

$$\det(\bar{\bar{A}})\det(\bar{\bar{A}}^T) = \det(\bar{\bar{I}}) = 1$$

since $\bar{\bar{A}}\bar{\bar{A}}^T = \bar{\bar{I}}$ and

$$\det(\bar{\bar{A}}\bar{\bar{B}}) = \det(\bar{\bar{A}})\det(\bar{\bar{B}})$$

$$\Rightarrow \quad \det^2(\bar{\bar{A}}) = 1$$

$\Rightarrow \det(\bar{\bar{A}}) = \pm 1$ for an orthogonal matrix.

Assume for now $\det(\bar{\bar{A}}) = 1$

$$\Rightarrow \quad a_{11}a_{22} - a_{12}a_{21} = 1$$

If $\bar{\bar{A}}$ is a real matrix then $0 \leq a_{11} \leq 1$. Choose $a_{11} = \cos\theta$ without loss of generality. Also we will get $a_{22}^2 = \cos^2\theta$ and $a_{21}^2 = a_{12}^2 = \sin^2\theta$.

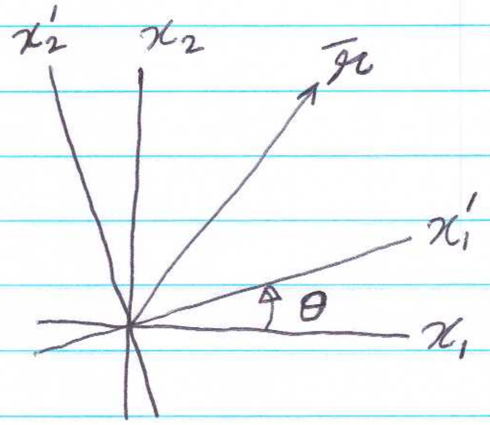
Now $\quad a_{11}a_{22} = 1 + a_{12}a_{21}$

$$\Rightarrow \quad \pm\cos^2\theta = 1 \pm \sin^2\theta$$

where we can choose $\pm$ independently on both sides. The only way this will hold is if we choose $a_{22} = a_{11}$ and $a_{21} = -a_{12} = -\sin\theta$

$$\therefore \bar{\bar{A}} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is just the rotation matrix for rotating an axis (XYZ) to $(X'Y'Z')$ by angle θ as shown.



This is a passive transformation.

Fig 4.4.

Note $x_3' \equiv x_3$, i.e. $Z' \equiv Z$.

So we get $\bar{r}' = \bar{\bar{A}}\bar{r}$

This means that we can get the components of $\bar{r}$ in the primed frame by operating $\bar{\bar{A}}$ on the column matrix of components of $\bar{r}$ in the original frame.

Alternatively it can mean keeping the axis fixed but rotating $\bar{r}$ to $\bar{r}'$ as shown, by an angle $-\theta$.

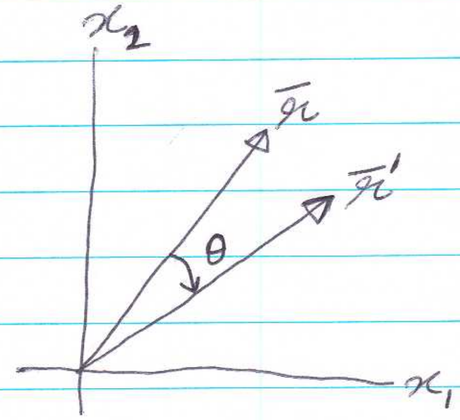


Fig. 4.5.

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This is an active transformation.

Some matrix properties :→

In general $\bar{\bar{A}}\,\bar{\bar{B}} \neq \bar{\bar{B}}\,\bar{\bar{A}}$

But $\bar{\bar{A}}(\bar{\bar{B}}\,\bar{\bar{C}}) = (\bar{\bar{A}}\,\bar{\bar{B}})\bar{\bar{C}}$ associativity

If $\bar{\bar{A}}\,\bar{\bar{A}}^{-1} = \bar{\bar{I}}$ then
$\bar{\bar{A}}^{-1}\bar{\bar{A}} = \bar{\bar{I}}$

By definition for an orthogonal matrix

$$\bar{\bar{A}}^{T}\bar{\bar{A}} = \bar{\bar{A}}\,\bar{\bar{A}}^{T} = \bar{\bar{I}}$$

$$\Rightarrow \bar{\bar{A}}^{T} = \bar{\bar{A}}^{-1}$$

$$(\bar{\bar{A}}\bar{\bar{B}})^{T} = \bar{\bar{B}}^{T}\bar{\bar{A}}^{T}$$

If for a square matrix
$\bar{\bar{A}} = \bar{\bar{A}}^{T}$ then it is defined to be a symmetric matrix.

If $\bar{\bar{A}}^T = -\bar{\bar{A}} \iff \bar{\bar{A}}$ is antisymmetric or skew symmetric. For such a matrix its diagonal elements are zero.

$$A_{ij} = -A_{ji} \Rightarrow A_{ii} = 0.$$ $\boxed{4-9}$

Consider the transformation $\bar{\bar{A}}$ which takes a column matrix $\bar{f}$ to $\bar{g}$

$$\Rightarrow \quad \bar{g} = \bar{\bar{A}}\bar{f}$$

Now ~~the~~ let the co-ordinate axes be changed $\ni$ they go to $\bar{\bar{B}}\bar{g}$ from $\bar{g}$,

$$\therefore \bar{\bar{B}}\bar{g} = \bar{\bar{B}}\bar{\bar{A}}\bar{f} = (\bar{\bar{B}}\bar{\bar{A}}\bar{\bar{B}}^{-1})\bar{\bar{B}}\bar{f} \rightarrow (4.40)$$

$\Rightarrow$ an operator $\bar{\bar{A}}$ in the old co-ordinate system becomes $\bar{\bar{B}}\bar{\bar{A}}\bar{\bar{B}}^{-1}$ in the new system.

$$\therefore \quad \bar{\bar{B}}\bar{g} = \bar{\bar{A}}'(\bar{\bar{B}}\bar{f})$$

$$\Rightarrow \quad \bar{\bar{A}}' = \bar{\bar{B}}\bar{\bar{A}}\bar{\bar{B}}^{-1} \longrightarrow (4.41)$$

Eq. (4.41) is called a similarity transformation.

$$\det(\bar{\bar{A}}') = \det(\bar{\bar{B}})\det(\bar{\bar{A}})\det(\bar{\bar{B}}^{-1})$$

$$\bar{\bar{B}}\,\bar{\bar{B}}^{-1} = \bar{\bar{I}}$$

$$\Rightarrow \det(\bar{\bar{B}})\det(\bar{\bar{B}}^{-1}) = \det(\bar{\bar{I}}) = 1$$

$$\Rightarrow \det(\bar{\bar{A}}') = \det(\bar{\bar{A}})$$

$\Rightarrow$ A similarity transformation leaves the determinant of $\bar{\bar{A}}$ unchanged.

$$\text{Trace}(\bar{\bar{A}}) \equiv Tr(\bar{\bar{A}}) \equiv \sum_i A_{ii}$$

Prove as an excercise that the similarity transformation (ST) leaves the trace unchanged.

$$Tr(\bar{\bar{A}}') \equiv Tr(\bar{\bar{B}}\bar{\bar{A}}\bar{\bar{B}}^{-1}) = Tr(\bar{\bar{A}})$$

Also if $\bar{\bar{A}}^T = \bar{\bar{A}}^{-1}$ then $\bar{\bar{A}}$ is an orthogonal transformation (OT) then

$$\det(\bar{\bar{A}}) = \pm 1$$

(OT)s with +1 value are called proper Ts and those with −1 values are called improper. Improper OTs do not correspond to physical rotations.

Euler Angles :→

In general we need 3 angular rotations to take a frame XYZ to X'Y'Z' as described below

Steps :→

① Rotate XYZ about Z by $+\phi$.

$$XYZ \xrightarrow[\left[\bar{\bar{D}} \text{ matrix}\right]]{} \xi\eta\zeta \equiv \xi\eta Z$$

② Rotate about ξ axis by $+\theta$

$$\xi\eta\zeta \xrightarrow[\left[\bar{\bar{C}} \text{ matrix}\right]]{} \xi'\eta'\zeta' \equiv \xi\eta'\zeta'$$

③ Rotate about ζ' axis by $+\psi$

$$\xi'\eta'\zeta' \xrightarrow[\left[\bar{\bar{B}} \text{ matrix}\right]]{} x'y'z'$$

These steps can be shown by matrix operations on column vectors ∋

$$\bar{\xi} = \bar{\bar{D}}\,\bar{x}$$

$$\xi' = \bar{\bar{C}}\,\xi \quad , \quad \bar{x}' = \bar{\bar{B}}\,\xi'$$

$$\bar{x}' = \bar{\bar{A}}\,\bar{x} \Rightarrow \bar{\bar{A}} = \bar{\bar{D}}\,\bar{\bar{C}}\,\bar{\bar{B}}$$

$$\bar{\bar{D}} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

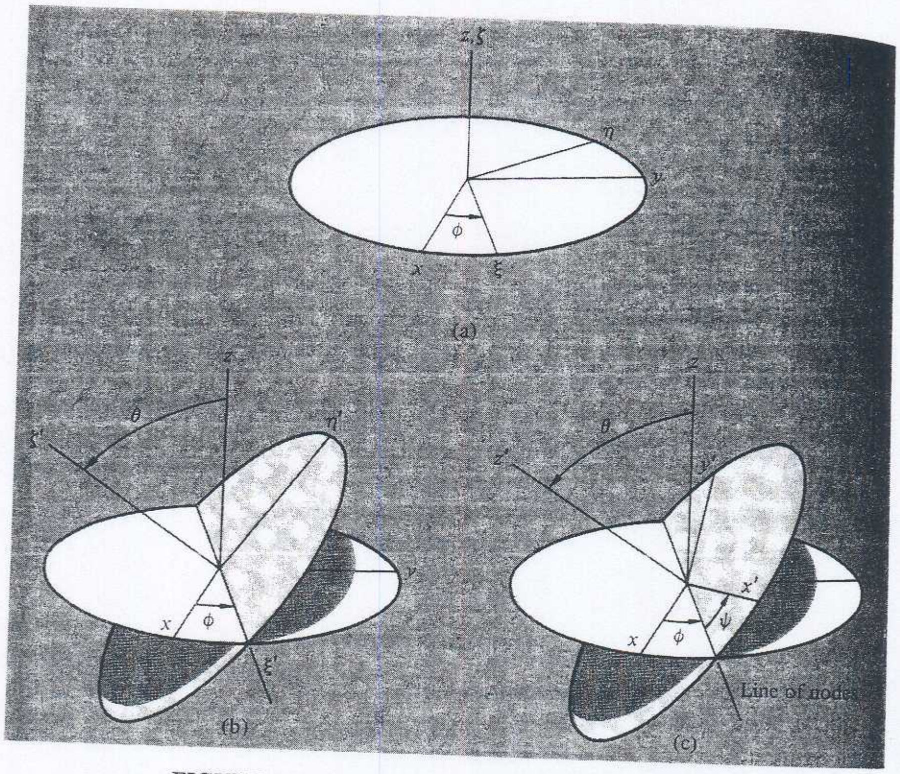

FIGURE 4.7 The rotations defining the Eulerian angles.

terclockwise by an angle ψ about the ζ' axis to produce the desired $x'y'z'$ system of axes. Figure 4.7 illustrates the various stages of the sequence. The Euler angles θ, ϕ, and ψ thus completely specify the orientation of the $x'y'z'$ system relative to the xyz and can therefore act as the three needed generalized coordinates.*

The elements of the complete transformation **A** can be obtained by writing the matrix as the triple product of the separate rotations, each of which has a relatively simple matrix form. Thus, the initial rotation about z can be described by a matrix **D**:

$$\boldsymbol{\xi} = \mathbf{D}\mathbf{x},$$

where $\boldsymbol{\xi}$ and $\mathbf{x}$ stand for column matrices. Similarly, the transformation from $\xi\eta\zeta$ to $\xi'\eta'\zeta'$ can be described by a matrix **C**,

*A number of minor variations will be found in the literature even within this convention. The differences are not very great, but they are often sufficient to frustrate easy comparison of the end formulae, such as the matrix elements. Greatest confusion, perhaps, arises from the occasional use of left-handed coordinate systems.

$$\bar{\bar{C}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

$$\bar{\bar{B}} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\bar{\bar{A}} = \left[\begin{array}{c|c|c} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{array}\right] \quad (4.46)$$

Note $\bar{x} = \bar{\bar{A}}^{-1}\bar{x}'$

$$\bar{\bar{A}}^{T} = \bar{\bar{A}}^{-1}$$

The temporal evolution of the orientation of a rigid body with time t may be denoted by a matrix $\bar{\bar{A}}(t)$.
Note $\bar{\bar{A}}(0) = \bar{\bar{I}}$.

Euler's Theorem :→ The general displacement of a rigid body with one fixed point is a rotation about some axis.

Proof :→ Consider a proper rotation matrix $\bar{\bar{A}}$, $\ni$ $\bar{\bar{A}}^{T} = \bar{\bar{A}}^{-1}$, $\det(\bar{\bar{A}}) = 1$.
We already have seen that this matrix

corresponds to a rotation about some axis.

Consider

$$(\bar{A}-\bar{\bar{I}})\bar{A}^T = \bar{A}\bar{A}^T - \bar{A}^T$$
$$= \bar{\bar{I}} - \bar{A}^T$$

$$\Rightarrow \det[(\bar{A}-\bar{\bar{I}})\cdot\bar{A}^T] = \det(\bar{\bar{I}}-\bar{A}^T)$$

$$\Rightarrow \det(\bar{\bar{I}}-\bar{A}^T) = \det(\bar{A}-\bar{\bar{I}})\det(\bar{A}^T)$$

$$\det(\bar{A}^T) = 1$$

$$\Rightarrow \det(\bar{\bar{I}}-\bar{A}^T) = \det(\bar{A}-\bar{\bar{I}})$$

Now $\det(\bar{\bar{I}}-\bar{A}^T) = \det(\bar{\bar{I}}^T-\bar{A}^T)$

$$= \det[(\bar{\bar{I}}-\bar{A})^T] = \det(\bar{\bar{I}}-\bar{A})$$

$$\Rightarrow \det(\bar{\bar{I}}-\bar{A})_{n\times n} = \det(\bar{A}-\bar{\bar{I}})_{n\times n}$$

n = dimensionality

For $\forall \bar{\bar{B}}_{n\times n}$,

$$\det(-\bar{\bar{B}}_{n\times n}) = (-1)^n \det(\bar{\bar{B}})$$

For $\bar{A}$, $n = 3$

$$\Rightarrow \det(\bar{\bar{I}}-\bar{A}) = (-1)^3 \det[-(\bar{\bar{I}}-\bar{A})]$$

$$\Rightarrow \det(\bar{A}-\bar{\bar{I}}) = 0$$

This is just the condition for the existence of an eigenvalue λ of $\bar{\bar{A}}$ $\ni \lambda = 1$

$$\bar{\bar{A}}\,\bar{e}_\lambda = \lambda\,\bar{e}_\lambda \quad \cancel{\text{with}}$$

$$\Rightarrow \left[\bar{\bar{A}} - \lambda \bar{\bar{\bar{I}}}\right] \bar{e}_\lambda = 0$$

$\Rightarrow$ for a solution $\bar{e}_\lambda \neq \bar{0}$

$$\det\left(\bar{\bar{A}} - \lambda \bar{\bar{\bar{I}}}\right) = 0 \quad \longrightarrow (4.52)$$

$\Rightarrow$ $\bar{\bar{A}}$ has one eigenvalue $\lambda = 1$

$$\Rightarrow \exists\, \bar{e}_1 \ni \bar{\bar{A}}\,\bar{e}_1 = \bar{e}_1$$

$\Rightarrow$ $\bar{e}_1$ is unaffected by $\bar{\bar{A}}$ a general ~~sub~~ displacement of a rigid body. [leaves a vector along an axis of rotation unaffected]

$\Rightarrow$ $\bar{\bar{A}}$ can be expressed as a rotation about an axis.

In general $\exists\, \lambda_1, \lambda_2, \lambda_3$ as solutions of (4.52). We proved $\lambda_1 = 1$

Consider a matrix

$$\bar{\bar{E}}_{3\times 3} = \left[(\bar{e}_{\lambda_1})_{3\times 1} \; (\bar{e}_{\lambda_2})_{3\times 1} \; (\bar{e}_{\lambda_3})_{3\times 1} \right]$$

$$\bar{\bar{A}}\,\bar{e}_\lambda = \lambda\,\bar{e}_\lambda$$

$$\therefore\ \left(\bar{\bar{A}}\,\bar{\bar{E}}\right)_{3\times 3} = \left[\lambda_1\left(\bar{e}_{\lambda_1}\right)_{3\times 1}\quad \lambda_2\left(\bar{e}_{\lambda_2}\right)_{3\times 1}\quad \lambda_3\left(\bar{e}_{\lambda_3}\right)_{3\times 1}\right]$$

$$= \bar{\bar{\lambda}}\,\bar{\bar{E}}$$

where $\bar{\bar{\lambda}}_{ij} \equiv \lambda_i \delta_{ij}$

$$\therefore\ \bar{\bar{E}}^{-1}\bar{\bar{A}}\,\bar{\bar{E}} = \bar{\bar{E}}^{-1}\bar{\bar{\lambda}}\,\bar{\bar{E}} = \bar{\bar{\lambda}}\,\bar{\bar{E}}^{-1}\bar{\bar{E}}$$

$$= \bar{\bar{\lambda}}$$

$$\Rightarrow\ \det\left(\bar{\bar{E}}^{-1}\bar{\bar{A}}\,\bar{\bar{E}}\right) = \det\left(\bar{\bar{A}}\right) = \det\left(\bar{\bar{\lambda}}\right)$$

$$= \lambda_1\lambda_2\lambda_3\ ,\quad \text{but } \det\left(\bar{\bar{A}}\right) = 1 = \lambda_1$$

$$\Rightarrow\ \lambda_2\lambda_3 = 1 .$$

Since $\bar{\bar{A}}$ is a real matrix

$\Rightarrow\ A^*_{ij} = A_{ij}$, where $*$ denotes complex conjugation

$\Rightarrow\ \lambda_2^* = \lambda_3$ since $\lambda_1 = 1 =$ real number.

$$\Rightarrow\ \lambda_2^*\lambda_2 = \lambda_3^*\lambda_3 = 1$$

$\Rightarrow$ ~~If~~ $\lambda_2 = e^{i\Phi}$ & $\lambda_3 = e^{-i\Phi}$, $\phi \in R$, Now $Tr(\bar{\bar{A}}) = Tr(\bar{\bar{\lambda}}) = \lambda_1 + \lambda_2 + \lambda_3$

$$= 1 + e^{i\Phi} + e^{-i\Phi} = 1 + 2\cos\Phi.$$

$\bar{\bar{A}}$ can now be transformed so that the general displacement is a rotation about the z axis so that

$$\bar{\bar{A}}' = \begin{bmatrix} \cos\Phi & \sin\Phi & 0 \\ -\sin\Phi & \cos\Phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Chasle's Theorem: → The most general displacement of a rigid body is a translation plus a rotation.

Finite rotations: → Consider rotations by angle $-\Phi$ of a vector $\bar{r}$ about

an axis, the unit vector along which is $\hat{n}$. Let the new vector after rotation is $\bar{r}'$. Note

$$\bar{r}' \cdot \hat{n} = \bar{r} \cdot \hat{n}$$

From Fig 4.8 we get then

$$\bar{r}' = (\bar{r}' \cdot \hat{n})\hat{n} + \bar{r}' - (\bar{r}' \cdot \hat{n})\hat{n}$$
$$= (\bar{r} \cdot \hat{n})\hat{n} + \bar{r}\cos\Phi - \hat{n}(\hat{n} \cdot \bar{r})\cos\Phi$$
$$+ (\bar{r} \times \hat{n})\sin\Phi$$

ters of the rotation—the angle of rotation and the direction cosines of the axis of rotation.

With the help of some simple vector algebra, we can derive such a representation. For this purpose, it is convenient to treat the transformation in its active sense, i.e., as one that rotates the vector in a fixed coordinate system (cf. Section 4.2). Recall that a counterclockwise rotation of the coordinate system then appears as a *clockwise* rotation of the vector. In Fig. 4.8(a) the initial position of the vector $\mathbf{r}$ is denoted by $\overrightarrow{OP}$ and the final position $\mathbf{r}'$ by $\overrightarrow{OQ}$, while the unit vector along the axis of rotation is denoted by $\mathbf{n}$. The distance between O and N has the magnitude $\mathbf{n} \cdot \mathbf{r}$, so that the vector $\overrightarrow{ON}$ can be written as $\mathbf{n}(\mathbf{n} \cdot \mathbf{r})$. Figure 4.8(b) sketches the vectors in the plane normal to the axis of rotation. The vector $\overrightarrow{NP}$ can be described also as $\mathbf{r} - \mathbf{n}(\mathbf{n} \cdot \mathbf{r})$, but its magnitude is the same as that of the vectors $\overrightarrow{NQ}$ and $\mathbf{r} \times \mathbf{n}$. To obtain the desired relation between $\mathbf{r}'$ and $\mathbf{r}$, we construct $\mathbf{r}'$ as the sum of three vectors:

$$\mathbf{r}' = \overrightarrow{ON} + \overrightarrow{NV} + \overrightarrow{VQ}$$

or

$$\mathbf{r}' = \mathbf{n}(\mathbf{n} \cdot \mathbf{r}) + [\mathbf{r} - \mathbf{n}(\mathbf{n} \cdot \mathbf{r})] \cos \Phi + (\mathbf{r} \times \mathbf{n}) \sin \Phi.$$

A slight rearrangement of terms leads to the final result:

$$\mathbf{r}' = \mathbf{r} \cos \Phi + \mathbf{n}(\mathbf{n} \cdot \mathbf{r})(1 - \cos \Phi) + (\mathbf{r} \times \mathbf{n}) \sin \Phi. \tag{4.62}$$

Equation (4.62) will be referred to as the *rotation formula*. Note that Eq. (4.62) holds for any rotation, no matter what its magnitude, and thus is a finite-rotation version (in a clockwise sense) of the description given in Section 2.6, for the change of a vector under infinitesimal rotation. (cf. also Section 4.8.)

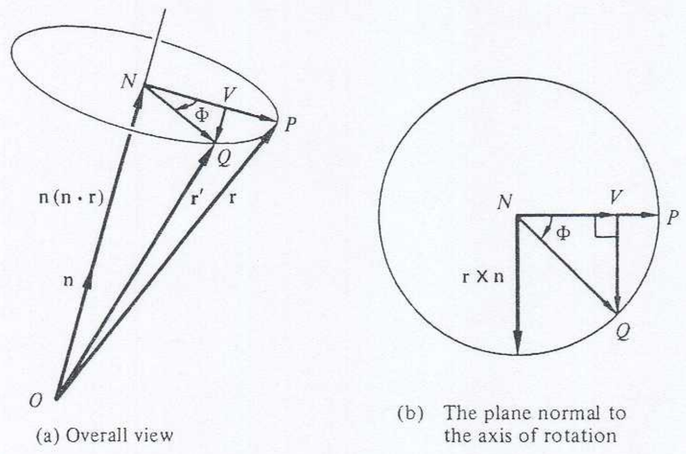

FIGURE 4.8 Vector diagrams for derivation of the rotation formula.

$$\Rightarrow \bar{r}' = \hat{n}(\hat{n}\cdot\bar{r}) + [\bar{r} - \hat{n}(\hat{n}\cdot\bar{r})]\cos\Phi + (\bar{r}\times\hat{n})\sin\Phi \longrightarrow \text{(4.62)}$$

We know $Tr.(\bar{\bar{A}}) = 1 + 2\cos\Phi$ where $\bar{\bar{A}}$ is this rotation by $-\Phi$ around $\hat{n}$

$$Tr.(\bar{\bar{A}}) = \cos\theta[1 + \cos(\phi+\psi)] + \cos(\phi+\psi)$$

from Eq. 4.46 from the Euler angles.

$$\Rightarrow 1 + 2\cos\Phi = \cos(\phi+\psi)[1+\cos\theta] + \cos\theta$$

$$\Rightarrow 2[1+\cos\Phi] = [1+\cos\theta][1+\cos(\phi+\psi)]$$

$$\Rightarrow \cos\left(\frac{\Phi}{2}\right) = \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\phi+\psi}{2}\right)$$

Infinitesimal rotations :→

Consider the inversion matrix defined as

$$\bar{\bar{S}} = -\bar{\bar{I}}, \text{ or } S_{ij} = -\delta_{ij}$$

$$\bar{\bar{S}}\cdot\bar{r} = -\bar{r}$$

$\Rightarrow$ $\bar{r}$ is a polar vector.

Any vector for which $\bar{\bar{S}}\cdot\bar{f} = -\bar{f}$ is a polar one.

Any vector $\bar{f}$ for which

$$\bar{\bar{S}} \cdot \bar{f} = +\bar{f}$$ is an axial vector.

$$\bar{\bar{S}} \cdot \bar{p} = -\bar{p}$$

$$\therefore \bar{\bar{S}} \cdot \bar{L} = \bar{\bar{S}} \cdot [\bar{r} \times \bar{p}] = \bar{L}$$

$\Rightarrow$ $\bar{L}$ is an axial vector

Consider an infinitesimal rotation $+d\Phi$

$$\Rightarrow \overline{dr} = \bar{r}' - \bar{r} = \bar{r}(\cos(d\Phi) - 1)$$

$$+ \hat{n}(\hat{n} \cdot \bar{r})[1 - \cos(d\Phi)] \div (\bar{r} \times \hat{n}) \sin(d\Phi)$$

$$\lim_{d\Phi \to 0} \cos(d\Phi) = 1 \quad \& \quad \sin(d\Phi) = d\Phi$$

$$\Rightarrow \overline{dr} = (\hat{n} \times \bar{r})\, d\Phi$$

Let $\overline{d\Omega} \equiv \hat{n}\, d\Phi$

$$\Rightarrow \overline{dr} = \overline{d\Omega} \times \bar{r}$$

Note $\overline{dr}$ is ~~an~~ polar vector and so is $\bar{r}$ $\Rightarrow$ $\overline{d\Omega}$ is an axial vector.

In matrix form we get

$$\bar{r}' = [\bar{\bar{I}} + \bar{\bar{\epsilon}}]\, \bar{r}$$

$$\Rightarrow \overline{dr} = \bar{r}' - \bar{r} = \overline{\overline{\epsilon}} \cdot \bar{r}$$

Since $\overline{\overline{A}} \equiv \overline{\overline{I}} + \overline{\overline{\epsilon}}$

and $\overline{\overline{A}}^{-1} \equiv \overline{\overline{I}} + \overline{\overline{k}}$

$$\Rightarrow (\overline{\overline{I}} + \overline{\overline{k}})(\overline{\overline{I}} + \overline{\overline{\epsilon}}) = \overline{\overline{I}}$$

$\Rightarrow \overline{\overline{k}} = -\overline{\overline{\epsilon}}$ to first order

$$\Rightarrow \overline{\overline{A}}^{-1} = \overline{\overline{I}} - \overline{\overline{\epsilon}}$$

$$\overline{\overline{A}}^{T} = \overline{\overline{A}}^{-1} \Rightarrow \overline{\overline{\epsilon}}^{T} = -\overline{\overline{\epsilon}}$$

$\Rightarrow \overline{\overline{\epsilon}}$ is antisymmetric

Also $\overline{\overline{\epsilon}} \equiv [\overline{d\Omega} \times]$

$$\Rightarrow \overline{\overline{\epsilon}} = \begin{bmatrix} 0 & -d\Omega_3 & d\Omega_2 \\ d\Omega_3 & 0 & -d\Omega_1 \\ -d\Omega_2 & d\Omega_1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} d\Phi \equiv \overline{\overline{N}} d\Phi$$

where $\overline{d\Omega} \equiv \hat{n}\, d\Phi$ and

$$\hat{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}.$$

$$\therefore \frac{d\bar{r}}{d\Phi} = \hat{n} \times \bar{r}$$

$$= \bar{\bar{N}} \cdot \bar{r}$$

Rate of change of a vector

Let subscipt $L \equiv S \equiv$ laboratory or space frame of reference

For any vector $\bar{G}$ then

$$(d\bar{G})_S = (d\bar{G})_B + (d\bar{G})_r$$

$(d\bar{G})_B =$ change in $\bar{G}$ as measured in a frame fixed to the rigid body.

$(d\bar{G})_r =$ infinitesimal change in $\bar{G}$ due to rotation of the rigid body.

$$\therefore (d\bar{G})_r = d\bar{\Omega} \times \bar{G}$$

$$\Rightarrow \left(\frac{d\bar{G}}{dt}\right)_S = \left(\frac{d\bar{G}}{dt}\right)_B + \frac{d\bar{\Omega}}{dt} \times \bar{G}$$

Let $\bar{\omega} \equiv \frac{d\bar{\Omega}}{dt}$

$\overline{\overline{E}}$ may also be expressed as

$$\overline{\overline{E}} = \sum_i n_i \overline{\overline{M}}_i \, d\phi$$

where $\overline{\overline{M}}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

$$\overline{\overline{M}}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad \& \quad \overline{\overline{M}}_3 = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The Lie brackets or commutators obey

$$\left[\overline{\overline{M}}_i, \overline{\overline{M}}_j\right] \equiv \overline{\overline{M}}_i \overline{\overline{M}}_j - \overline{\overline{M}}_j \overline{\overline{M}}_i = \sum_k \epsilon_{ijk} \overline{\overline{M}}_k$$

where $\epsilon_{ijk} = 0$, if either 2 indices are equal

$= +1$, if i, j, k are cyclic

$= -1$, " i, j, k are anticyclic

as an operator equation.

As a rigid body rotates about some fixed point on ~~its~~, the Euler angles (ϕ, θ, ψ) also change.

Since ϕ is a rotation about Z axis (or Z_S axis).

we get $(\bar{\omega}_\phi)_B = \bar{\bar{A}}(\bar{\omega}_\phi)_S$

where $(\bar{\omega}_\phi)_S \equiv \dot{\phi}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Using $\bar{\bar{A}}$ of Eq. 4.46 we get

$$(\bar{\omega}_\phi)_B = \begin{bmatrix} \dot{\phi}\sin\theta\sin\psi \\ \dot{\phi}\sin\theta\cos\psi \\ \dot{\phi}\cos\theta \end{bmatrix}$$

Since ψ is a rotation about the z' (or Z_B) axis

$$(\bar{\omega}_\psi)_B = \dot{\psi}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

θ is a rotation around the ξ' axis and remembering that

$$\bar{x}' = \bar{\bar{B}}\,\xi' \quad \text{i.e. } (\bar{x})_B = \bar{\bar{B}}\,\xi'$$

$$\Rightarrow (\bar{\omega}_\theta)_B = \bar{\bar{B}}(\bar{\omega}_\theta)_{\xi'\eta'\zeta'}$$

$$(\bar{\omega}_\theta)_{\xi'\eta'\zeta'} = \dot{\theta}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow (\bar{\omega}_\theta)_B = \dot{\theta}\begin{bmatrix} \cos\psi \\ -\sin\psi \\ 0 \end{bmatrix}$$

$$\Rightarrow \bar{\omega}_B = \bar{A}(\bar{\omega}_\phi)_S + (\bar{\omega}_\psi)_B + \bar{\bar{B}}(\bar{\omega}_\theta)_{\xi'\eta'\zeta'}$$

$$= \begin{bmatrix} \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \dot{\phi}\cos\theta + \dot{\psi} \end{bmatrix}$$

The Coriolis Effect :→

Applying (4.86) to the position vector $\bar{r}$ of a particle we get

$$\bar{V}_s \equiv \left(\frac{d\bar{r}_s}{dt}\right)_s = \bar{V}_B + \bar{\omega} \times \bar{r}_s$$

Applying it to $\bar{V}_s$ we get

$$\bar{a}_s \equiv \left(\frac{d\bar{V}_s}{dt}\right)_s = \bar{a}_B + \bar{\omega} \times \bar{V}_B + \bar{\omega} \times \left(\frac{d\bar{r}_s}{dt}\right)_s$$

$$= \bar{a}_B + (\bar{\omega} \times \bar{V}_B) + \bar{\omega} \times \left[\bar{V}_B + \bar{\omega} \times \bar{r}_s\right]$$

$$= \bar{a}_B + 2(\bar{\omega} \times \bar{V}_B) + \bar{\omega} \times (\bar{\omega} \times \bar{r}_s)$$

where we assumed a constant ~~consta~~ rotational ~~vel~~ velocity $\bar{\omega}$

i.e. $\dot{\bar{\omega}} = 0$.

∴ in the space frame

$$\bar{F} = m\bar{a}_s \quad \text{becomes}$$

$$\bar{F} - 2m(\bar{\omega} \times \bar{V}_B) - m\bar{\omega} \times (\bar{\omega} \times \bar{r}_s) = m\bar{a}_B$$

∋ $\bar{F}_{eff} = \bar{F} - 2m(\bar{\omega} \times \bar{V}_B) - m\bar{\omega} \times (\bar{\omega} \times \bar{r}_s)$,