If $L = L(\{q_i\}, \{\dot{q}_i\}, t)$

satisfies Lagrange's equations

then

$$L'(\{q_i\}, \{\dot{q}_i\}, t) = L(\{q_i\}, \{\dot{q}_i\}, t) + \frac{dF}{dt}$$

also satisfies them, where

$$F = F(\{q_i\}, t).$$

Also note

$$T = \sum_i \frac{m_i}{2} \bar{v}_i^{\,2} = \sum_i \left\{ \frac{m_i}{2} \left[\frac{\partial \bar{r}_i}{\partial t} + \sum_j \left(\frac{\partial \bar{r}_i}{\partial q_j} \right) \dot{q}_j \right] \right.$$

$$\left. \times \left[\frac{\partial \bar{r}_i}{\partial t} + \sum_k \left(\frac{\partial \bar{r}_i}{\partial q_k} \right) \dot{q}_k \right] \right\}$$

$$= T_0 + T_1 + T_2$$

where $T_0 \equiv M_0 \equiv \sum_i \frac{m_i}{2} \left(\frac{\partial \bar{r}_i}{\partial t} \right)^2$

$$T_1 = \sum_{i,j} m_i \left(\frac{\partial \bar{r}_i}{\partial t}\right) \cdot \left(\frac{\partial \bar{r}_i}{\partial q_j}\right) \dot{q}_j$$

and

$$T_2 = \sum_{i,j,k} \frac{m_i}{2} \left(\frac{\partial \bar{r}_i}{\partial q_j}\right) \cdot \left(\frac{\partial \bar{r}_i}{\partial q_k}\right) \dot{q}_j \dot{q}_k$$

If $\bar{r}_i$ has no explicit time dependence then

$$\frac{\partial \bar{r}_i}{\partial t} = 0$$

$$\Rightarrow \quad T_1 = T_0 = 0$$

$$T = T_2 \quad \text{is quadratic in } \{\dot{q}_j\}.$$

Will encounter again in small oscillations.

D'Alembert's principle is a "differential principle"

The instantaneous configuration of a system is given by n generalized co-ordinates $\{q_i\}$, $i=1,\dots n$.

$\{q_i\}$ together form one point

in an n-dimensional space. This space of different configurations is called configuration space.

Then Hamilton's principle states that

$$I = \int_{t_1}^{t_2} L\,dt \quad \text{is stationary}$$

for the actual path followed by a system in configuration space

i.e. $\delta I = 0$

where $L \equiv T - V$.

$$L = L(\{q_i\}, \{\dot{q}_i\}, t)$$

This is an "integral principle" and is true for monogenic systems.

Monogenic systems are those for which all forces except the workless constraint forces are derivable from a potential. The potential may be a function of $\{q_i\}$, $\{\dot{q}_i\}$ and time t.

Crudely we define a functional as a number that depends on the value of a function at more than one point.

$y(x) = f(x)$ defines y as a function of x. But y where

$$y(x) = \int_0^x f(x')\,dx',$$

is a function of x which is also a functional of f since it depends on all values of f between 0 and x. We denote this as

$$y = y[f(x)].$$

Hence $I = I[L(\{q_i\}, \{\dot{q}_i\}, t)]$

Often we encounter functionals

$$F[h(p)] \equiv \int_{p_1}^{p_2} f(h(p), h'(p), p)\, dp$$

where p_1 & p_2 are fixed parameters

p is a ~~paramet~~ parameter.
We need to find a function

$h_s = h_s(p)$, $\ni$ $F[h(p)]$ is

stationary

$$\ni \left.\frac{\delta F[h(p)]}{\delta h(p)}\right|_{h=h_s(p)} = 0 \,, \quad \forall p \in [p_1, p_2]$$

where $\frac{\delta}{\delta h(p)}$ defines a functional derivative
It is defined in general as

$$\frac{\delta \Phi[h(x)]}{\delta h(y)} = \lim_{\epsilon \to 0} \left\{ \frac{\Phi[h(x) + \epsilon\delta(x-y) - \Phi[h(x)]}{\epsilon} \right\}$$

where $\Phi[h(x)]$ is a functional

of $h(x)$. $\delta(x-y)$ is the Dirac delta function $\ni \lim_{x \to y} \delta(x-y) \to \infty$

$$\delta(x-y) = 0, \quad \forall x \neq y$$

$$\& \quad \int_{y-a}^{y+a} \delta(x-y)\, dx = 1, \quad \forall a > 0.$$

As a special case let

(i) $\Phi[h(x)] = h(x)$

$$\Rightarrow \frac{\delta h(x)}{\delta h(y)} = \delta(x-y).$$

(ii) $\Phi[h(x)] = f(h(x))$ a mere function

$$\Rightarrow \frac{\delta f(h(x))}{\delta h(y)} = \left[\frac{df(h(x))}{dh(x)}\right] \times \frac{\delta h(x)}{\delta h(y)} = f'\delta(x-y)$$

(iii) $\Phi[h(x)] = f(g(h(x)))$ a mere function of function

$$\Rightarrow \frac{\delta \Phi[h(x)]}{\delta h(y)} = f'g'\,\delta(x-y)$$

where $f' \equiv \dfrac{df(g)}{dg}$, $g' \equiv \dfrac{dg(h)}{dh}$

Now consider

$$F[h(p)] = \int_{p_1}^{p_2} f(h(p), h'(p), p)\,dp$$

$$\therefore \frac{\delta F[h(p)]}{\delta h(s)} = \int_{p_1}^{p_2} dp \left\{ \frac{\delta f(h(p), h'(p), p)}{\delta h(s)} \right\}$$

$$= \int_{p_1}^{p_2} dp \left\{ \frac{\partial f(h(p), h'(p), p)}{\partial h(p)} \times \frac{\delta h(p)}{\delta h(s)} \right.$$

$$\left. + \frac{\partial f(h(p), h'(p), p)}{\partial h'(p)} \times \frac{\delta h'(p)}{\delta h(s)} \right\}$$

$$= \int_{p_1}^{p_2} dp \left\{ \frac{\partial f(h(p), h'(p), p)}{\partial h(p)} \times \delta(p-s) \right.$$

$$\left. + \frac{\partial f(h(p), h'(p), p)}{\partial h'(p)} \times \frac{d}{dp}\left(\frac{\delta h(p)}{\delta h(s)} \right) \right\}$$

$$= \frac{\partial f(h(s), h'(s), s)}{\partial h(s)} + \int_{p_1}^{p_2} dp \left\{ \frac{d}{dp}\left(\frac{\partial f}{\partial h'(p)} \times \delta(p-s) \right) \right\}$$

$$- \int_{p_1}^{p_2} dp \left\{ \left[\frac{d}{dp}\left(\frac{\partial f}{\partial h'(p)} \right) \right] \times \delta(p-s) \right\}$$

$$= \frac{\partial f(h(s), h'(s), s)}{\partial h(s)} - \frac{d}{ds}\left(\frac{\partial f(h(s), h'(s), s)}{\partial h'(s)} \right)$$

$$\therefore \frac{\delta F[h(p)]}{\delta h(s)} = 0$$

$$\Rightarrow \frac{\partial f}{\partial h} - \frac{d}{ds}\left(\frac{\partial f}{\partial h'}\right) = 0 \quad , \forall s \in [p_1, p_2]$$

Let $F \equiv I$, $\Leftrightarrow f \equiv L \Leftrightarrow p \equiv t$

$\Leftrightarrow h \equiv q$ then we get

Lagrange's equations.

We can generalize to ~~m~~ n variables $\{q_i\}$ to get

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

$\Rightarrow$ Hamilton's principle

$\Leftrightarrow$ Lagrange's eqns.

Also called Euler-Lagrange or Euler's equations.

Consider ~~me~~ now that $\exists$ m holonomic constraints of the form

$$f_\alpha(\bar{r}_1, \bar{r}_2, \ldots, t) = 0 \quad , \alpha = 1, 2, \ldots m.$$

In

$$\delta I = 0 \quad \text{we now}$$

replace $\quad L$ by $L + \sum_{\alpha=1}^{m} \lambda_\alpha f_\alpha$

$\lambda_\alpha \equiv$ Lagrange undetermined multipliers.

Note that f_α are not functions of $\dot{q}_i$ so

$$\frac{\partial f_\alpha}{\partial \dot{q}_i} = 0.$$

Now in analogy to previous derivation we will reach a step

$$\frac{\delta I[q_i(t)]}{\delta q_j(s)} = \sum_{i=1}^{n} \int_{t_1}^{t_2} dt \left\{ \frac{\partial L}{\partial q_j(s)} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j(s)}\right) + \sum_{\alpha=1}^{m} \lambda_\alpha \frac{\partial f_\alpha}{\partial q_j(s)} \right\} \frac{\delta q_i(t)}{\delta q_j(s)}$$

Note: Earlier we used

$$\frac{\delta q_i(t)}{\delta q_j(s)} = \delta_{ij}\, \delta(t-s)$$

where δ_{ij} = Kronecker delta function

$= 1 \quad , \forall i = j$

$= 0 \quad , \forall i \neq j.$

We cannot use this now since $\{q_i\}$, $i=1,\ldots n$ are not independent $\because \exists\ m$ constraints.

$\therefore$ we demand that $\forall j=1,\ldots m$ we choose the set $\{\lambda_\alpha\}$, $\alpha=1,\ldots m$ to satisfy

$$\frac{\partial L}{\partial q_j(t)} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j(t)}\right) + \sum_{\alpha=1}^{m} \lambda_\alpha \frac{\partial f_\alpha}{\partial q_j(t)} = 0 \quad \rightarrow (2.23)$$

$\forall j = 1, \ldots m.$

Now the remaining set $\{q_i\}$, $i=n-m+1, n-m+2, \ldots n$ can be treated as having all q_i independent. We can then use

$$\frac{\delta q_i(t)}{\delta q_j(s)} = \delta_{ij}\,\delta(t-s), \quad \forall\ i,j \in [n-m+1, n]$$

We again get the equations (2.23) but now $\forall j \in [n-m+1, n]$

$\Rightarrow$ (2.23) are valid $\forall j \in [1, n]$.