Constraints :→

A constraint that can be expressed as

$$f(\bar{r}_1, \bar{r}_2 \ldots, t) = 0$$ is called

a holonomic constraint.

Constraints explicitly depending on time are called rheonomous. Those not explicitly dependent on time are called scleronomous constraints.

E.g. Scleronomous ≡ Bead sliding on a rigid curved wire fixed in space
Rheonomous ≡ If wire moves in some known fashion.

Generalized coordinates are used to
(i) eliminate constraint forces
(ii) to make independent coordinates from dependent ones.

N particles ⇒ 3N degrees of freedom if there are no constraints.
k holonomic constraints ⇒ 3N−k degrees of freedom.

D' Alembert's principle

$$\sum_i (\bar{F}_i - \dot{\bar{p}}_i) \cdot \delta r_i = 0 \longrightarrow (1.44)$$

Let the constraint forces be $\bar{f}_i$ and the applied forces be $\bar{F}_i^a$ so that

$$\bar{F}_i = \bar{F}_i^a + \bar{f}_i$$

$\bar{f}_i$ do no virtual work

$$\Rightarrow \bar{f}_i \cdot \overline{\delta r_i} = 0$$

$$\Rightarrow \sum_i \bar{F}_i^a \cdot \overline{\delta r_i} = 0$$

Let $\bar{r}_i = \bar{r}_i(q_1, q_2, \cdots q_n, t)$

$\{q_i\}$ are the generalized coordinates,

$$\therefore \overline{\delta r_i} = \sum_j \frac{\partial \bar{r}_i}{\partial q_j} \delta q_j \longrightarrow (1.47)$$

$$\therefore \sum_i \bar{F}_i^a \cdot \overline{\delta r_i} = \sum_j Q_j \delta q_j \longrightarrow (1.48)$$

where

$$Q_j \equiv \sum_i \bar{F}_i^a \cdot \frac{\partial \bar{r}_i}{\partial q_j} \longrightarrow (1.49)$$

Note $\frac{d}{dt}\left(\frac{\partial \bar{r}_i}{\partial q_j}\right) = \frac{\partial}{\partial q_j}\left(\frac{d\bar{r}_i}{dt}\right)$

$$= \frac{\partial \bar{V}_i}{\partial q_j}$$

$$\therefore \frac{d}{dt}\left(\frac{\partial \bar{r}_i}{\partial q_j}\right) = \frac{\partial \bar{V}_i}{\partial q_j} \longrightarrow 1.505$$

Also $\bar{V}_i \equiv \frac{d\bar{r}_i}{dt} = \sum_k \left(\frac{\partial \bar{r}_i}{\partial q_k}\right) \dot{q}_k + \frac{\partial \bar{r}_i}{\partial t}$

where $\dot{q}_k \equiv \frac{dq_k}{dt}$

$$\therefore \frac{\partial \bar{V}_i}{\partial \dot{q}_j} = \sum_k \left(\frac{\partial \bar{r}_i}{\partial q_k}\right) \delta_{kj}$$

where $\delta_{kj} \equiv$ Kronecker delta function

$$\ni \delta_{kj} = 1, \quad \forall k = j$$
$$= 0 \quad \text{otherwise}$$

$$\therefore \frac{\partial \bar{V}_i}{\partial \dot{q}_j} = \frac{\partial \bar{r}_i}{\partial q_j} \longrightarrow 1.51$$

Kinetic energy T_i is related to $\bar{p}_i$

$$T_i \equiv \frac{\bar{p}_i^{\,2}}{2m_i}, \quad T \equiv \sum_i T_i$$

Now $\dot{\bar{p}}_i \cdot \overline{\delta r_i}$

$$= \sum_j \left(\frac{d}{dt}\bar{p}_i\right) \cdot \left(\frac{\partial \bar{r}_i}{\partial q_j}\right) \delta q_j$$

$$= \sum_j \left\{ \frac{d}{dt}\left[\bar{p}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j}\right] - \bar{p}_i \cdot \frac{d}{dt}\left(\frac{\partial \bar{r}_i}{\partial q_j}\right) \right\} \delta q_j$$

[Using 1.505 & 1.51]

$$= \sum_j \left\{ \frac{d}{dt}\left[\bar{p}_i \cdot \frac{\partial \bar{V}_i}{\partial \dot{q}_j}\right] - \bar{p}_i \cdot \frac{\partial \bar{V}_i}{\partial q_j} \right\} \delta q_j$$

[Using $\bar{V}_i = \bar{p}_i / m_i$

$$= \sum_j \left\{ \frac{d}{dt}\left[\bar{p}_i \cdot \frac{\partial}{\partial \dot{q}_j}(\bar{p}_i / m_i)\right] - \bar{p}_i \cdot \frac{\partial}{\partial q_j}\left(\frac{\bar{p}_i}{m_i}\right) \right\} \delta q_j$$

$$= \sum_j \left\{ \frac{d}{dt}\left(\frac{\partial T_i}{\partial \dot{q}_j}\right) - \frac{\partial T_i}{\partial q_j} \right\} \delta q_j = \dot{\bar{p}}_i \cdot \overline{\delta r_i}$$

$$\therefore \sum_j \left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j \right\} \delta q_j = 0$$

Since $\{q_i\}$ are independent coordinates we get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \longrightarrow (1.53)$$

Suppose $\bar{F}_i^a = -\bar{\nabla}_i V(\bar{r}_1, \bar{r}_2 \ldots \bar{r}_N)$

Then $Q_j = \sum_i \bar{F}_i^a \cdot \frac{\partial \bar{r}_i}{\partial q_j}$

$$\boxed{= \cancel{\sum_i \bar{\nabla}_i \bar{F}_i^a \cdot \frac{\partial \bar{r}_i}{\partial q_j}} =}$$

$$= -\sum_i (\bar{\nabla}_i V) \cdot \frac{\partial \bar{r}_i}{\partial q_j} = \frac{\partial V}{\partial q_j}$$

$$\therefore Q_j = \frac{\partial V}{\partial q_j} \longrightarrow (1.54)$$

Since V does not depend on any velocities $\frac{\partial V}{\partial \dot{q}_j} = 0$, $\forall j$.

$\therefore$ (1.53) can be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \longrightarrow (1.57)$$

where $L \equiv T - V$ is defined as the Lagrangian of the system.

Now suppose as an alternative to 1·54 we have

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) \longrightarrow \text{1·58}$$

where $U = U(q_1, q_2, \cdots q_n, \dot{q}_1, \dot{q}_2, \cdots \dot{q}_n, t)$

$$\equiv U(\{q_i\}, \{\dot{q}_i\}, t)$$

Then 1·58 with 1·53 again leads to Lagrange's equations 1·57 with

$$L \equiv T - U.$$

Consider $U = q\,\phi(\bar{r}, t) - q\,\bar{v}\cdot\bar{A}(\bar{r}, t)$

where q is the charge on a particle at $\bar{r}$ moving with velocity $\bar{v}$.

$$\therefore\; L = \frac{mv^2}{2} - q\phi + q\,\bar{A}\cdot\bar{v}$$

Let $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$ represent a set of orthogonal unit vectors along the orthogonal set of co-ordinates axes.

such as $\hat{X}, \hat{Y}, \hat{Z}$ axes.

$$\therefore \quad q_i \equiv \bar{r}\cdot\hat{e}_i \quad , \quad i = 1, 2, 3 \text{ and}$$

$$\dot{q}_i \equiv \bar{V}\cdot\hat{e}_i \quad .$$

$$\therefore \frac{\partial L}{\partial q_i} = \frac{\partial L}{\partial(\bar{r}\cdot\hat{e}_i)} = -q\frac{\partial\phi}{\partial(\bar{r}\cdot\hat{e}_i)} + q\sum_{k=1}^{3}(\bar{V}\cdot\hat{e}_k)\frac{\partial}{\partial(\bar{r}\cdot\hat{e}_i)}(\bar{A}\cdot\hat{e}_k)$$

$$\frac{\partial L}{\partial(\bar{V}\cdot\hat{e}_i)} = m(\bar{V}\cdot\hat{e}_i) + q(\bar{A}\cdot\hat{e}_i)$$

$$\therefore \frac{d}{dt}\left[\frac{\partial L}{\partial(\bar{V}\cdot\hat{e}_i)}\right] = m\frac{d(\bar{V}\cdot\hat{e}_i)}{dt} + q\frac{\partial(\bar{A}\cdot\hat{e}_i)}{\partial t} + q\sum_{k=1}^{3}(\bar{V}\cdot\hat{e}_k)\frac{\partial}{\partial(\bar{r}\cdot\hat{e}_k)}(\bar{A}\cdot\hat{e}_i)$$

$\therefore$ putting all these terms together we get

$$m\frac{d}{dt}(\bar{V}\cdot\hat{e}_i) = -q\frac{\partial\phi}{\partial(\bar{r}\cdot\hat{e}_i)} - q\frac{\partial(\bar{A}\cdot\hat{e}_i)}{\partial t} + q\sum_{k=1}^{3}(\bar{V}\cdot\hat{e}_k)\left[\frac{\partial(\bar{A}\cdot\hat{e}_k)}{\partial(\bar{r}\cdot\hat{e}_i)} - \frac{\partial(\bar{A}\cdot\hat{e}_i)}{\partial(\bar{r}\cdot\hat{e}_k)}\right], \ \forall i = 1, 2, 3$$

If we identify

$$\bar{E} \equiv -\bar{\nabla}\phi - \frac{\partial \bar{A}}{\partial t}$$

and $\bar{B} \equiv \bar{\nabla} \times \bar{A}$ as the

electromagnetic fields then the 3 Lagrange's equations maybe combined as

$$\bar{F} \equiv m\bar{a} = q\bar{E} + q(\bar{v} \times \bar{B}),$$

giving the Lorentz force on a particle moving in an electromagnetic field.

Now consider as an alternative to (1.54)

$$Q_j = -\frac{\partial V}{\partial q_j} + Q_j' \quad \text{where}$$

$$Q_j' \equiv \sum_i \bar{F}_i^{nc} \cdot \frac{\partial \bar{r}_i}{\partial q_j} \longrightarrow (1.665)$$

where $\bar{F}_i^a \equiv -\bar{\nabla}_i V + \bar{F}_i^{nc}$

$nc \equiv$ non-conservative.

Then (1.57) is modified to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = \mathcal{Q}_j' \longrightarrow (1.575)$$

where $L \equiv T - V$ as before.

This form is useful for dissipative forces.

Consider Rayleigh's dissipation function

$$\mathcal{F} \equiv \frac{1}{2}\sum_{p=1}^{N}\sum_{k=1}^{3} (\bar{k}\cdot\hat{e}_i)(\bar{V}_p\cdot\hat{e}_i)^2$$

where $\bar{k}$ is a constant vector

Let

$$\bar{F}_p^{nc} = -\bar{\nabla}_{V_p}\mathcal{F}$$

Note p stands for the particle index and i for components.

Then (1.665) becomes

$$\mathcal{Q}_j' = \sum_p \bar{F}_p^{nc}\cdot\frac{\partial \bar{r}_p}{\partial q_j} = -\sum_p\left(\bar{\nabla}_{V_p}\mathcal{F}\right)\cdot\frac{\partial \bar{r}_p}{\partial q_j}$$

$$= -\left(\frac{\partial \mathcal{F}}{\partial \dot{q}_j}\right)$$, where we use (1.51)

Hence we get

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} + \frac{\partial \mathcal{F}}{\partial \dot{q}_j} = 0$$

The work done by the system of particles against $\bar{F}_p^{nc}$ is

$$dW_{nc} = -\sum_p \bar{F}_p^{nc} \cdot d\bar{r}_p$$

$$= -\sum_p \left[-(\bar{\nabla}_{v_p} \mathcal{F}) \cdot \bar{v}_p \, dt \right]$$

$$= 2\mathcal{F} \, dt$$

Hence $2\mathcal{F} = \dfrac{dW_{nc}}{dt}$ = power dissipated by the system.

Study section 1·6 of the text as home-work. Ask questions next time.