Division of Engineering
Brown University

EN3: Introduction to Engineering

Teach Yourself Vectors

APPENDIX II. ADVANCED TOPIC: Non-Cartesian vector components

This material will not be covered in EN3 vector proficiency exams, and will not be used during the statics part of the course. It is provided here for completeness – you will need to know and use this material in EN4 and in EN51. You might like to come back to this tutorial again when you need it.

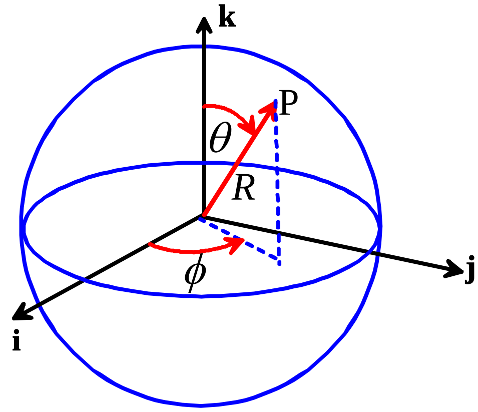
Now that we understand vector components better, we can go a step further and introduce two new schemes for expressing an arbitrary vector as components, called spherical-polar and cylindrical-polar coordinate systems.

II.1 Spherical-polar coordinates

In problems involving astrodynamics or geodesic survey, we often need to specify positions, velocities, etc relative to the center of the earth, or relative to an observer located on the earth's surface. With the tools we have developed so far, we would establish an appropriate {**i,j,k**} basis, select an origin at the center of the earth, and then write position vector as $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. The problem with this approach is that it's very difficult to visualize just where a point like ($5013\mathbf{i} + 1123\mathbf{j}+6073\mathbf{k}$) km is, in relation to a point on the earth's surface. So as you already know, we abandon this approach and instead specify position as latitude, longitude and height above the earth's surface. When you do this, you are in fact using a spherical-polar coordinate system.

Before you read any further, please make sure you recall what latitude and longitude are! The equator has zero latitude; the Greenwich meridian has zero longitude. With the north pole `up', lines of constant latitude are horizontal stripes around the earth. Lines of constant longitude are vertical stripes. Its easy to remember this: horizontal stripes make you look **fat** -> **lat**itude.

II.1.1 Specifying points in spherical-polar coordinates



To specify points in space using spherical-polar coordinates, we first choose two convenient, mutually perpendicular reference directions (**i** and **k** in the picture). When we specify position on the Earth's surface for example, we might choose **k** to point from the center of the earth towards the North Pole, and choose **i** to point from the center of the earth towards the intersection of the equator (which has zero degrees latitude) and the Greenwich Meridian (which has zero degrees longitude, by definition).

Then, each point P in space is identified by three numbers, R, θ, ϕ shown in the picture above. **These are not components of a vector**.

In words:

R is the distance of P from the origin

θ is the angle between the **k** direction and OP

ϕ is the angle between the **i** direction and the projection of OP onto a plane through O normal to **k** (got that? Try saying it fast…)

To attach a familiar physical significance to these variables

R is the distance of a point from the center of the earth

θ is (90 degrees- the latitude of a point) for points North of the equator, or (90 degrees+ latitude) for points South of the equator

ϕ is the longitude of a point for Easterly longitudes, or (360 degrees-longitude) for Westerly longitudes.

By convention, we choose $R \geq 0$ **,** $0 \leq \theta \leq 180^o$ **and** $0 \leq \phi \leq 360^o$

Exercises

II.1 Using the spherical-polar coordinate system described in the preceding section, specify the points

(a) The position of the geographic North and South poles
(b) The position of Providence, Rhode Island (try http://www.bcca.org/misc/qiblih/latlong_us.html
(c) The position of Greenwich, England (try searching the web for its latitude)
(d) The position of a geo-stationary satellite (which must lie in the equatorial plane) on the same longitude as Washington DC. Try seaching the web to find the distance of a geo-stationary satellite from the earth's center.

The earth's radius is about 6378km.

II.2 Using elementary trigonometry, derive expressions for the components of the position vector of point P relative to O in terms of R, θ, ϕ , expressing your answer as components in the {**i,j,k**} basis.

II.1.2 Converting between Cartesian and Spherical-Polar representations of points

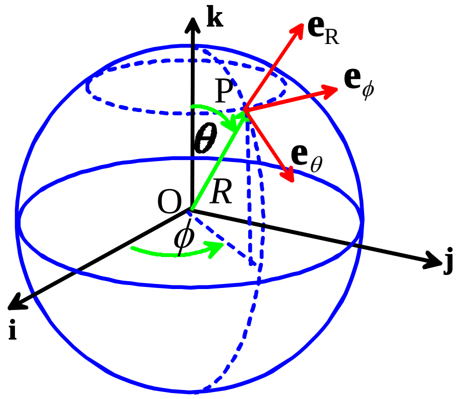
When we use a Cartesian basis, we identify points in space by specifying the components of their position vector relative to the origin (*x*,*y*,*z*). When we use a spherical-polar coordinate system, we locate points by specifying their spherical-polar coordinates R, θ, ϕ .

The formulas below relate the two representations. They are derived using basic trigonometry (see problem 7.2)

$$x = R\sin\theta\cos\phi \qquad R = \sqrt{x^2 + y^2 + z^2}$$
$$y = R\sin\theta\sin\phi \qquad \theta = \cos^{-1} z/R$$
$$z = R\cos\theta \qquad \phi = \tan^{-1} y/x$$

II.1.3 Spherical-Polar representation of vectors

When we work with vectors in spherical-polar coordinates, we abandon the {**i,j,k**} basis. Instead, we specify vectors as components in the $\{\mathbf{e}_R, \mathbf{e}_\theta, \mathbf{e}_\phi\}$ basis shown in the figure below



The basis is different for each point *P*. In words

- $\mathbf{e}_R$ points along OP
- $\mathbf{e}_\theta$ is tangent to a line of constant longitude through P
- $\mathbf{e}_\phi$ is tangent to a line of constant latitude through P.

For our example of specifying points on the Earth's surface, you can visualize the basis vectors like this. Suppose you stand at a point P on the Earths surface. Relative to you,

$\mathbf{e}_R$ points vertically upwards

$\mathbf{e}_\theta$ points due South

$\mathbf{e}_\phi$ points due East.

Notice that the basis vectors depend on where you are standing,

You can also visualize the directions as follows. To see the direction of $\mathbf{e}_R$, keep θ and ϕ fixed, and increase *R*. P is moving parallel to $\mathbf{e}_R$. To see the direction of $\mathbf{e}_\theta$, keep *R* and ϕ fixed, and increase θ . P now moves parallel to $\mathbf{e}_\theta$. To see the direction of $\mathbf{e}_\phi$, keep *R* and θ fixed, and increase ϕ . P now moves parallel to $\mathbf{e}_\phi$. Mathematically, this concept can be expressed as follows. Let **r** be the position vector of P. Then

$$\mathbf{e}_R = \frac{1}{\left|\frac{\partial \mathbf{r}}{\partial R}\right|}\frac{\partial \mathbf{r}}{\partial R} \qquad \mathbf{e}_\theta = \frac{1}{\left|\frac{\partial \mathbf{r}}{\partial \theta}\right|}\frac{\partial \mathbf{r}}{\partial \theta} \qquad \mathbf{e}_\phi = \frac{1}{\left|\frac{\partial \mathbf{r}}{\partial \phi}\right|}\frac{\partial \mathbf{r}}{\partial \phi}$$

A basis with these properties is called the `natural basis' for the coordinate system. For a spherical coordinate system, the natural basis happens to be an orthonormal basis (basis vectors are mutually perpendicular and form a right handed triad)

Exercises

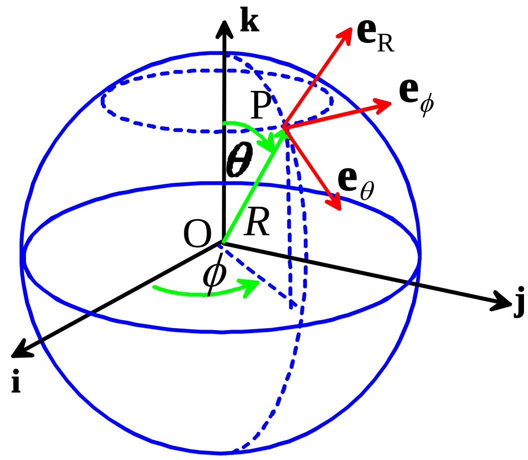
II.3 Consider a spherical-polar coordinate system intended to specify position relative to the Earth's center, as discussed in the preceding sections. Express the following quantities as components in the natural basis for this coordinate system

(a) The position vector of point P
(b) A 15 mile-per hour Southerly wind at P
(c) A 25 mile-per-hour North-Westerly wind at P

II.4 Suppose you stand at the intersection of the Greenwich meridian and the Equator. Let $\{\mathbf{e}_R, \mathbf{e}_\theta, \mathbf{e}_\phi\}$ be the basis associated with your position. Express the following quantities in this basis
(a) The position vector of the North Pole relative to the center of the Earth
(b) The position vector of the North Pole relative to you
(c) The direction you move if you jump up and down

(Bonus points if you know which country you are in at the intersection of the Greenwich meridian and the Equator!)

II.1.4 Converting vectors between Cartesian and Spherical-Polar bases



Let $\mathbf{a} = a_R \mathbf{e}_R + a_\theta \mathbf{e}_\theta + a_\phi \mathbf{e}_\phi$ be a vector. Find a formula for the components of **a** in the basis {**i**,**j**,**k**}, i.e. find a_x, a_y, a_z such that $\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}$

It is easiest to do the transformation by expressing each basis vector $\{\mathbf{e}_R, \mathbf{e}_\theta, \mathbf{e}_\phi\}$ as components in {**i**,**j**,**k**}, and then substituting. To do this, recall that $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, recall also the conversion

$$x = R\sin\theta\cos\phi$$
$$y = R\sin\theta\sin\phi$$
$$z = R\cos\theta$$

and finally recall that by definition

$$\mathbf{e}_R = \frac{1}{\left|\frac{\partial \mathbf{r}}{\partial R}\right|}\frac{\partial \mathbf{r}}{\partial R} \qquad \mathbf{e}_\theta = \frac{1}{\left|\frac{\partial \mathbf{r}}{\partial \theta}\right|}\frac{\partial \mathbf{r}}{\partial \theta} \qquad \mathbf{e}_\phi = \frac{1}{\left|\frac{\partial \mathbf{r}}{\partial \phi}\right|}\frac{\partial \mathbf{r}}{\partial \phi}$$

Hence, substituting for *x*,*y*,*z* and differentiating

$$\mathbf{r} = R\sin\theta\cos\phi\mathbf{i} + R\sin\theta\sin\phi\mathbf{j} + R\cos\theta\mathbf{k}$$
$$\Rightarrow \frac{\partial \mathbf{r}}{\partial R} = \sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k}$$

Conveniently we find that $\left|\frac{\partial \mathbf{r}}{\partial R}\right| = 1$ (check this for yourself, recalling the trig simplification $\sin^2 A + \cos^2 A = 1$)

Therefore

$$\mathbf{e}_R = \sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k}$$

Similarly

$$\frac{\partial \mathbf{r}}{\partial \theta} = R\cos\theta\cos\phi\mathbf{i} + R\cos\theta\sin\phi\mathbf{j} - R\sin\theta\mathbf{k}$$

and $\left|\frac{\partial \mathbf{r}}{\partial \theta}\right| = R$, so that

$$\mathbf{e}_\theta = \cos\theta\cos\phi\mathbf{i} + \cos\theta\sin\phi\mathbf{j} - \sin\theta\mathbf{k}$$

The third basis vector follows as,

$$\frac{\partial \mathbf{r}}{\partial \phi} = -R\sin\theta\sin\phi\mathbf{i} + R\sin\theta\cos\phi\mathbf{j}$$

and $\left|\frac{\partial \mathbf{r}}{\partial \phi}\right| = R\sin\theta$, so that

$$\mathbf{e}_\phi = -\sin\phi\mathbf{i} + \cos\phi\mathbf{j}$$

Finally, substituting

$$\mathbf{a} = a_R[\sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k}]$$
$$+ a_\theta[\cos\theta\cos\phi\mathbf{i} + \cos\theta\sin\phi\mathbf{j} - \sin\theta\mathbf{k}]$$
$$+ a_\phi[-\sin\phi\mathbf{i} + \cos\phi\mathbf{j}]$$

Collecting terms in **i**, **j** and **k**, we see that

$$a_x = \sin\theta\cos\phi a_R + \cos\theta\cos\phi a_\theta - \sin\phi a_\phi$$
$$a_y = \sin\theta\sin\phi a_R + \cos\theta\sin\phi a_\theta + \cos\phi a_\phi$$
$$a_z = \cos\theta a_R - \sin\theta a_\theta$$

If you like matrices, this transformation can be expressed as

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} a_R \\ a_\theta \\ a_\phi \end{bmatrix}$$

Conversely, let $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$. Find components $\mathbf{a} = a_R\mathbf{e}_R + a_\theta\mathbf{e}_\theta + a_\phi\mathbf{e}_\phi$

This time, we can use the formal approach presented in Sect. 6. We have

$$\mathbf{a} = a_R\mathbf{e}_R + a_\theta\mathbf{e}_\theta + a_\phi\mathbf{e}_\phi = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$
$$\Rightarrow \mathbf{a}\cdot\mathbf{e}_R = a_R = a_x\mathbf{i}\cdot\mathbf{e}_R + a_y\mathbf{j}\cdot\mathbf{e}_R + a_z\mathbf{k}\cdot\mathbf{e}_R$$

(where we have used $\mathbf{e}_\theta\cdot\mathbf{e}_R = \mathbf{e}_\phi\cdot\mathbf{e}_R = 0$). Recall that

$$\mathbf{e}_R = \sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k}$$
$$\Rightarrow \mathbf{i}\cdot\mathbf{e}_R = \sin\theta\cos\phi \qquad \mathbf{j}\cdot\mathbf{e}_R = \sin\theta\sin\phi \qquad \mathbf{k}\cdot\mathbf{e}_R = \cos\theta$$

Substituting, we get

$$a_R = \sin\theta\cos\phi a_x + \sin\theta\sin\phi a_y + \cos\theta a_z$$

Proceeding in exactly the same way for the other two components

$$a_\theta = \cos\theta\cos\phi a_x + \cos\theta\sin\phi a_y - \sin\theta a_z$$
$$a_\phi = -\sin\phi a_x + \cos\phi a_y$$

In matrix form

$$\begin{bmatrix} a_R \\ a_\theta \\ a_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\varphi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

(Comparing this result to the transformation from spherical to rectangular coordinates, we notice that the matrices involved in the transformation have a neat property – for each matrix, its inverse is equal to its transpose)

Exercises

II.5 The position vector of a point in spherical-polar coordinates is $\mathbf{r} = R\mathbf{e}_R$. Using the basis change formulas, check that you get the correct expression for the components of the position vector in the **{i,j,k}** basis.

II.6 Suppose you are located in Providence, Rhode Island (now there's a stretch). Let $\{\mathbf{e}_R, \mathbf{e}_\theta, \mathbf{e}_\phi\}$ be the basis associated with your position. Express the following quantities in this basis

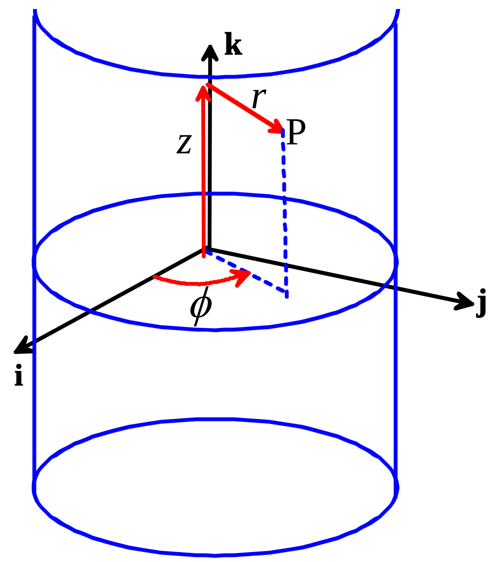
(a) The position vector of the North Pole relative to the center of the Earth
(b) The position vector of the North Pole relative to you
(c) The direction you move if you jump up and down (presumably in frustration, by this point)

II.2 Cylindrical-Polar Coordinates

For Freudian reasons, no doubt, many engineering problems involve objects with cylindrical geometry (carbon nanotubes, drive shafts, toilet paper rolls, electromagnetic coils…). Problems of this type are best solved using cylindrical-polar coordinates.

Once you've mastered spherical-polar coordinates it's quite easy to get up to speed on the cylindrical-polar coordinate system. Below, we summarize the important results, without extensive derivations (which will conveniently leave us with a good source of exam problems…)

II.2.1 Locating points in cylindrical-polar coordinates



To specify the location of a point in cylindrical-polar coordinates, we choose an origin at some point on the axis of the cylinder, select **k** to be parallel to the axis of the cylinder, and choose a convenient direction for the basis vector **i**, as shown in the picture. We then use the three numbers r,ϕ,z to locate a point inside the cylinder, as shown in the picture.

In words

r is the radial distance of P from the axis of the cylinder

ϕ is the angle between the **i** direction and the projection of OP onto the **i,j** plane

z is the length of the projection of OP on the axis of the cylinder.

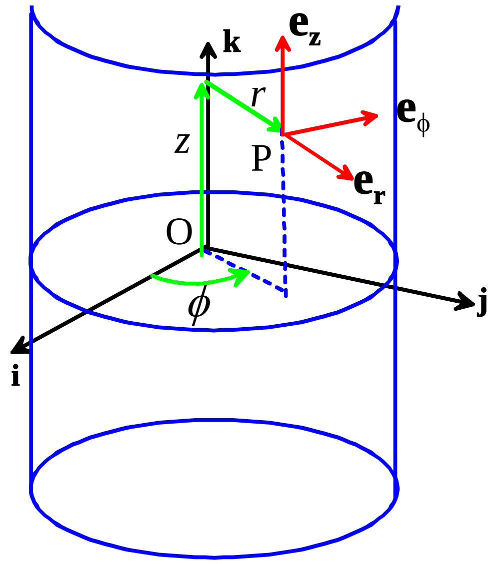
By convention *r>0* and $0 \le \phi \le 360^O$

II.2.2 Converting between cylindrical polar and rectangular cartesian coordinates

The formulas below convert from cartesian (*x,y,z*) coordinates to cylindrical polar r,ϕ,z coordinates and back again

$$x = r\cos\phi \qquad r = \sqrt{x^2+y^2}$$
$$y = r\sin\phi \qquad \phi = \tan^{-1} y/x$$
$$z = z \qquad z = z$$

II.2.3 Cylindrical-polar representation of vectors



When using cylindrical-polar coordinates, all vectors are expressed as components in the basis $\{\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_z\}$ shown. In words

- $\mathbf{e}_r$ is a unit vector normal to the cylinder at P
- $\mathbf{e}_\phi$ is a unit vector circumferential to the cylinder at P, chosen to make $\{\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_z\}$ a right handed triad
- $\mathbf{e}_z$ is parallel to the **k** vector.

You will see that the position vector of point P would be expressed as

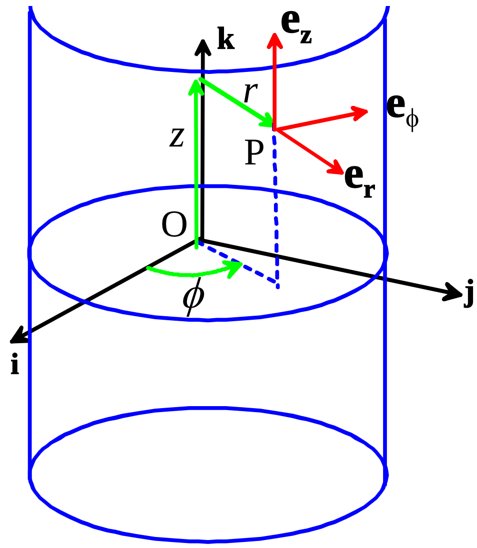
$$\mathbf{r} = r\mathbf{e}_r + z\mathbf{e}_z = r\cos\phi\mathbf{i} + r\sin\phi\mathbf{j} + z\mathbf{k}$$

Note also that the basis vectors are intentionally chosen to satisfy

$$\mathbf{e}_r = \frac{1}{\left|\frac{\partial \mathbf{r}}{\partial r}\right|}\frac{\partial \mathbf{r}}{\partial r} \qquad \mathbf{e}_\phi = \frac{1}{\left|\frac{\partial \mathbf{r}}{\partial \phi}\right|}\frac{\partial \mathbf{r}}{\partial \phi} \qquad \mathbf{e}_z = \frac{1}{\left|\frac{\partial \mathbf{r}}{\partial z}\right|}\frac{\partial \mathbf{r}}{\partial z}$$

and are therefore the natural basis for the coordinate system.

II.2.4 Converting vectors between cylindrical and cartesian bases



Let $\mathbf{a} = a_r\mathbf{e}_r + a_\phi\mathbf{e}_\phi + a_z\mathbf{e}_z$ be a vector, expressed as components in $\{\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_z\}$. It is straightforward to show that the components of **a** in {**i,j,k**} ($\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$) are

$$a_x = a_r \cos\phi - a_\phi \sin\phi$$
$$a_y = a_r \sin\phi + a_\phi \cos\phi$$
$$a_z = a_z$$

As a matrix

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_r \\ a_\phi \\ a_z \end{bmatrix}$$

The reverse of this transformation is

$$a_r = a_x \cos\phi + a_y \sin\phi$$
$$a_\phi = -a_x \sin\phi + a_y \cos\phi$$
$$a_z = a_z$$

In matrix form

$$\begin{bmatrix} a_r \\ a_\phi \\ a_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$