Final Examination for PHYS 4310/5310, Spring 2005

1. A state of three identical particles each of integer spin, is made of the spatial part and the spin part. The spatial part is to be made of either $\Psi_S(1,2,3)$ or $\Psi_A(1,2,3)$. The spin part is made of either $\chi_S(1,2,3)$ or $\chi_A(1,2,3)$. The subscripts S and A stand for symmetric and anti-symmetric respectively. Construct all possible total states of the particles from these spatial and spin parts. (2 points)

2. Two identical particles occupy exactly the same spin states. Therefore the symmetry of the spatial part of the state alone determines the symmetry of the entire state of the two particles. The spatial parts are made of two orthonormal single particle states $\Psi_a(x)$ and $\Psi_b(x)$, where the subscripts a and b denote the two different states. Define $\langle x\rangle_a \equiv \int_{-\infty}^{\infty} |\Psi_a(x)|^2 x dx$ and $\langle x\rangle_b \equiv \int_{-\infty}^{\infty} |\Psi_b(x)|^2 x dx$. In what follows express all wavefunctions in terms of $\Psi_a(x)$ and $\Psi_b(x)$ with appropriate subscripts 1 or 2 on x. Express all expectation values in terms of $\langle x\rangle_a$ or $\langle x\rangle_b$.

(a) Write an expression for the spatial part of the wavefunction $\Psi_F(x_1, x_2)$ if the two particles are fermions. Find $\langle(x_1 - x_2)\rangle_F$. (3 points)

(b) Write an expression for the spatial part of the wavefunction $\Psi_B(x_1, x_2)$ if the two particles are bosons. Find $\langle(x_1 - x_2)\rangle_B$. (3 points)

(c) Write an expression for the spatial part of the wavefunction $\Psi_D(x_1, x_2)$ if the two particles are distinguishable. Find $\langle(x_1 - x_2)\rangle_D$. (3 points)

3. For a system of N distinguishable particles the number of ways to obtain a configuration $\{N_n, n = 1, 2, ...\infty\}$ is given by

$$Q = (N!)\Pi_{n=1}^{\infty}\left[\frac{d_n^{N_n}}{N_n!}\right] \tag{1}$$

The derivation of this formula assumes that any number of particles can occupy a particular state with energy E_n. Instead we now restrict the derivation so that a maximum of 1 particle can occupy a state with energy E_n.

(a) What is the relationship between d_n and N_n now? (1 point)

(b) Only one of the three terms needs to be replaced by this change, either $(N!)$ or $(d_n^{N_n})$ or

$(N_n!)$. Which one is it? (1 point)

(c) Does this problem resemble another problem of indistiguishable particles. If so what type of particles are they? (1 point)

(d) Using answer to part (c) as a hint replace the appropriate term in the formula and write the new expression for Q. (2 points)

4. Assume that an atom contributes q free electrons to a continuous periodic one dimensional solid. Assume further that electrons have a spin quantum number s = 3/2, instead of the usual s = 1/2.

(a) How many of these electrons can have the spatial part of their wavefunctions to be identical. (1 point)

(b) If q = 8, would this solid be a conductor, insulator or semi-conductor? (1 point)

(c) If q = 13, would this solid be a conductor, insulator or semi-conductor? (1 point)

5. The energy per unit volume, per unit frequency, for an electromagnetic field in equilibrium at temperature T is given by

$$\frac{dE(\omega)}{V} = \frac{\hbar\omega^3 d\omega}{\pi^2 c^3(-1+\exp((\hbar\omega)/(k_B T)))}, \tag{2}$$

c is the speed of light in free space, k_B is Boltzmann's constant, ω is the angular frequency of the field, $\hbar$ is Planck's constant divided by 2π.

(a) Derive the scaling of the total energy density E/V of this electromagnetic field in the form

$$E/V = \hbar^\alpha c^\beta (k_B)^\gamma T^\delta G, \tag{3}$$

where G is a dimensionless numerical constant. You need not find the numerical constant G. (2 point)

(b) Find the values of the exponents α, β, γ, and δ. (1 point)

6. A particle in one dimension, is in the ground state of a potential $V(x) = -\alpha\delta(x)$, where $\alpha > 0$ and $\delta(x)$ is the Dirac delta function.

(a) Write (you need not derive) the ground state wavefunction $\Psi(x,t)$ of the particle. (1 point)

(b) Find the expectation value of the Hamiltonian $\langle \hat{\mathbf{H}} \rangle$. and hence the expectation value of the kinetic energy $\langle \hat{\mathbf{T}} \rangle$ in this state. (3 points)

7. A particle has a normalized wavefunction given by $\Psi(r,\theta,\phi,0) = \{R(r)[Y_3^{-2}(\theta,\phi) + Y_2^1(\theta,\phi)]\}/\sqrt{2}$. It is given that $\int_0^\infty |rR(r)|^2 dr = 1$. Two operators are defined as $\hat{\mathbf{A}}_\pm \equiv \hat{\mathbf{L}}_x \pm \hat{\mathbf{L}}_y$, in terms of the x and y components of the angular momemtum operator $\hat{\vec{\mathbf{L}}}$.

(a) Express the commutator $[\hat{\mathbf{A}}_+, \hat{\mathbf{A}}_-]$ in terms of $[\hat{\mathbf{L}}_x, \hat{\mathbf{L}}_y]$. (1 point)

(b) Express the result in (a) as $c_x\hat{\mathbf{L}}_x + c_y\hat{\mathbf{L}}_y + c_z\hat{\mathbf{L}}_z$. Find the coefficients c_x, c_y, and c_z (1 point)

(c) Find a lower limit on the product of uncertainties $\sigma_{A_+}\sigma_{A_-}$ in the given state. (2 points)