Analytic Solution to HO.

We want to solve TISE for

\[ V(x) = \frac{m\omega^2 x^2}{2} \]

\[ \hat{H} \psi_E = E \psi_E \text{ becomes} \]

\[ \frac{d^2 \psi_E(x)}{dx^2} = \frac{2m}{\hbar^2} \left[ \frac{m\omega^2 x^2}{2} - E \right] \psi_E(x) \quad \rightarrow (2.70) \]

Let \( y = \left( \frac{m\omega}{\hbar} \right)^{1/2} x \). Note book uses \( \xi \) for y.

\[ \Rightarrow \frac{d^2 \psi_E(y)}{dy^2} = (y^2 - K) \psi_E(y) \]

where \( K = \frac{2E}{\hbar^2 \omega} \)

We now look at approximate solution at \( y^2 \approx K \)

\[ \Rightarrow \psi_E''(y) \approx y^2 \psi_E(y) \]

\[ \Rightarrow \psi_E(y) \approx Ae^{-y^{3/2}} + Be^{y^{3/2}} \]

The term \( e^{y^{3/2}} \to \infty \) as \( y \to \infty \) and will not be normalizable. So we try a solution

\[ \psi_E(y) = h(y) e^{-y^{3/2}} \]
Putting this into Eq. (2.70) gives

\[ h''(y) - 2y h'(y) + (K-1) h = 0 \quad \rightarrow (2.78) \]

We now try a series solution

\[ h(y) = a_0 + a_1 y + a_2 y^2 + \ldots = \sum_{j=0}^{\infty} a_j y^j \]

\[ \Rightarrow h'(y) = \sum_{j=0}^{\infty} j a_j y^{j-1} \quad \text{and} \quad h''(y) = \sum_{j=0}^{\infty} j(j-1) a_j y^{j-2} \]

\[ \Rightarrow h''(y) = \sum_{j=0}^{\infty} \frac{(j+2)(j+1)}{j+2} a_{j+2} y^j \]

Using these series in Eq. (2.78) gives

\[ \sum_{j=0}^{\infty} y^j \left[ (j+2)(j+1) a_{j+2} - 2j a_j + (K-1) a_j \right] = 0 \]

For this to hold true \( \forall y \in \mathbb{R} \)

\[ (j+2)(j+1) a_{j+2} + (K-1-2j) a_j = 0 \]

\[ \Rightarrow a_{j+2} = \left[ \frac{1+2j-K}{(j+2)(j+1)} \right] a_j \quad \rightarrow (2.81) \]

This is a recursion formula for \( a_j \).

Note that if we know \( a_0 \) and \( a_1 \), then all higher order coefficients \( a_j \), \( j \geq 1 \) are known. Thus \( h(y) \) and hence \( y(y) \) is completely defined by just two constants \( a_0 \) and \( a_1 \), as expected for a second order differential equation.
Now we observe that
\[ a_{j+2} = \frac{2a_j}{j} \]  
from Eq. (2.81)

\[ a_j \approx a_{j-2} \left(\frac{j-2}{j^2}\right) \left(\frac{j-4}{j^2}\right) \]

\[ a_j \approx \frac{c}{(j/2)!} \]

\[ h(y) \approx c \sum \frac{y^j}{j!} \]

\[ h(y) \approx c \sum \frac{y^j}{j!} \approx ce^{-y^2} \]

\[ y(y) \approx (ce^{-y^2}) e^{y^2/2} \approx ce^{y^2/2} \]

\[ y(y) \to \infty \text{ as } y \to \infty. \text{ This is not a good solution.} \]

\( E_n = \frac{(n+1)}{2} \) or \( E_n = \frac{(n+1)}{2} \) \( \times \omega \)

The only way that will happen is if

\[ \exists \text{ an integer } n \exists \]

\[ K = 2n + 1, \ n \geq 10 \]

\[ E_n = \frac{(n+1)}{2} \] or \( E_n = \frac{(n+1)}{2} \) \( \times \omega \)

Note that this is the same energy spectrum we got earlier using \( \hat{a}_\pm \)
Now \( a_{n+2} = a_{n+1} = a_{n-1} = 0 \). However, this says nothing about \( a_{n-3}, a_{n-5}, \ldots \) and \( a_{n+3}, a_{n+5}, \ldots \).

- If \( n \) is odd, we need to choose \( a_0 = 0 \) and if \( n \) is even then \( a_0 = 0 \).

\[ h_n(y) = h_n(y) = n^{\text{th}} \text{ order polynomial} \]

- If \( n \) is odd, \( h_n(y) = -h_n(-y) \)

\[ \Rightarrow \Psi_n(y) = h_n(y) e^{-y^2/2} = -\Psi_n(-y) \]

- If \( n \) is even, \( \Psi_n(y) = h_n(y) e^{-y^2/2} = \Psi_n(-y) \)

\[ h_n(y) = a_0 H_n(y), \quad \forall n \text{ even} \]

\[ = a_n H_n(y), \quad \forall n \text{ odd} \]

\( H_n(y) \) are the Hermite polynomials defined by Eq. (2.8)

\[ H_0(y) = 1, \quad H_1(y) = 2y, \quad H_2(y) = 4y^2 - 2 \]

Now we can determine \( a_0 \) and \( a_n \) from normalization of \( \Psi_n(y) \) so we get

\[ \Psi_n(x) = (m \omega)^{1/4} \frac{H_n(y) e^{-y^2/2}}{\sqrt{2^n (n!)}} \text{, where } y = (m \omega)^{1/4} x \]