Linear Algebra

A vector space consists of a set of vectors $[|\alpha\rangle, |\beta\rangle, |\gamma\rangle \dots]$ together with a set of scalars $(a, b, c, \dots)$ which is closed under two operations

(1) Addition

$$|\alpha\rangle + |\beta\rangle = \text{another vector} \quad \forall |\alpha\rangle, |\beta\rangle \in S$$

(2) Scalar Multiplication

$$a|\alpha\rangle = \text{also a vector } \forall a, |\alpha\rangle.$$

$\forall \equiv$ "for all", $\equiv$ means "is equivalent to" or it means "is defined as"

Addition is commutative

$$\ni \quad |\alpha\rangle + |\beta\rangle = |\beta\rangle + |\alpha\rangle$$

and associative

$$|\alpha\rangle + (|\beta\rangle + |\gamma\rangle) = (|\alpha\rangle + |\beta\rangle) + |\gamma\rangle$$

$\exists \equiv$ "there exists"

There exists a zero or null vector such that

$$|\alpha\rangle + |0\rangle = |\alpha\rangle \ , \ \forall |\alpha\rangle \in S$$

$\ni \equiv$ "such that", $\in \equiv$ "belongs to"

We will in general call the set of vectors in the vector space **S**.
The scalars will be complex numbers i.e. their set is **C**. The set of real numbers will be called R.
Any scalar a can be expressed as

$$a \equiv a_r + i a_i \ , \text{ where } i \equiv \sqrt{-1}$$

$$a_r, a_i \in R.$$

$$a \equiv r e^{i\theta} \equiv r[\cos\theta + i \sin\theta]$$

$$\Longleftrightarrow \quad a_r \equiv r\cos\theta, \quad a_i \equiv r \sin\theta$$

$$r, \theta \in R$$

$$\theta \equiv \arctan\left[\frac{a_i}{a_r}\right]$$

$$r = \sqrt{a_r^2 + a_i^2} \ , \ r \geq 0.$$

$$|a| \equiv r \ , \ a^* \equiv a_r - i a_i$$

$$|a|^2 \equiv a a^* \equiv a^* a$$

$\forall |\alpha\rangle \in S$, $\exists$ an additive inverse of $|\alpha\rangle$ denoted by $-|\alpha\rangle \ni |\alpha\rangle - |\alpha\rangle = |0\rangle$

Scalar multiplication is ~~so~~ distributive with respect to vector addition

$$\Rightarrow \quad a\left(|\alpha\rangle + |\beta\rangle\right) = a|\alpha\rangle + a|\beta\rangle$$

and with respect to scalar addition

$$(a+b)|\alpha\rangle = a|\alpha\rangle + b|\alpha\rangle$$

It is also associative wrt scalar multiplication

$$a\left(b|\alpha\rangle\right) = (ab)|\alpha\rangle$$

wrt ≡ "with respect to".

Also $\quad 0|\alpha\rangle = |0\rangle$ and $1|\alpha\rangle = |\alpha\rangle$.

A linear combination of vectors is of the form $a|\alpha\rangle + b|\beta\rangle + c|\gamma\rangle + \cdots$.

Two vectors $|\alpha\rangle$ and $|\beta\rangle$ are defined to be linearly independent iff

the equation $\quad a|\alpha\rangle + b|\beta\rangle = |0\rangle$

has a unique solution $a = b = 0$ for the scalars a and b.

iff ≡ "if and only if".

Consider a set of m vectors $|e_i\rangle$

$i = 1, 2, \ldots m$. They are said to be linearly independent iff the equation

$$\sum_{i=1}^{m} a_i |e_i\rangle = |0\rangle \longrightarrow \text{NI}$$

has the unique solution $a_i = 0$, $\forall i = 1, \ldots m$.

The maximum value of m is called the dimension of the space n.
I.e. n is the maximum number of linearly independent vectors that can be found. A set of n linearly independent vectors is called a basis because any vector $|\alpha\rangle \in S$ can be expressed as

$$|\alpha\rangle = \sum_{i=1}^{n} a_i |e_i\rangle$$

So set $\{e_i, i = 1, \ldots n\}$ obeying NI $\ni$

$a_i = 0$, $\forall i = 1, \ldots n$ is called a basis.

It is said to span the space. Such a set is also called complete.
Vectors in S such as $|\alpha\rangle$ are called kets. For every ket $|\alpha\rangle$ there exists another vector denoted by $\langle\alpha|$ in the

space dual to S called S_d.

$$|\alpha\rangle \in S \Longleftrightarrow \langle\alpha| \in S_d$$

$A \Longrightarrow B \equiv$ "Statement A implies Statement B"

$A \Longleftrightarrow B \equiv$ "Statement A implies Statement B and Statement B implies Statement A".

An inner product between two vectors $|\alpha\rangle$ and $|\beta\rangle$ is denoted by $\langle\alpha|\beta\rangle$ and it belongs to C

I.e. $\langle\alpha|\beta\rangle \in C$.

For every equation with kets we have an equation with bras. We obtain it by replacing all kets with bras and all scalars by their complex conjugates.

$$\therefore \quad |\alpha\rangle = a|\beta\rangle + c|\gamma\rangle$$

$$\Longleftrightarrow \quad \langle\alpha| = \langle\beta|a^* + \langle\gamma|c^*$$

$\therefore$ if $d = \langle\delta|\omega\rangle$ then

$$d^* = \langle\omega|\delta\rangle, \text{ i.e. } \langle\delta|\omega\rangle^* = \langle\omega|\delta\rangle$$

and $\langle \delta | \omega \rangle = \langle \omega | \delta \rangle^*$

Also inner products have the following properties

$$\langle \alpha | \alpha \rangle \geq 0 \,, \quad \langle \alpha | \alpha \rangle = 0 \Leftrightarrow | \alpha \rangle = | 0 \rangle$$

$$\langle \alpha | \left[b | \beta \rangle + c | \gamma \rangle \right] = b \langle \alpha | \beta \rangle + c \langle \alpha | \gamma \rangle$$

The norm of a vector is defined as

$$\| \, | \alpha \rangle \| \equiv \left| \sqrt{\langle \alpha | \alpha \rangle} \right|$$

A unit vector is one whose norm is 1

$|\alpha\rangle$ is a unit vector $\Leftrightarrow \langle \alpha | \alpha \rangle = 1$.

A set of vectors $|\alpha_i\rangle$, $i = 1, \dots m$ is orthogonal (perpendicular) iff

$$\langle \alpha_i | \alpha_j \rangle = 0 \quad , \quad \forall i \neq j.$$

Furthermore if $\langle \alpha_i | \alpha_i \rangle = 1$, $\forall i = 1, \dots m$ then the set is called orthonormal. If $m = n =$ dimension of S then the set is called an orthonormal basis.

We define the Kronecker delta function or discrete delta function by

$$\delta_{ij} = 1 \quad ; \quad \forall i = j$$
$$\phantom{\delta_{ij}} = 0 \quad ; \quad \forall i \neq j$$

e.g. $\delta_{1,2} = 0$, $\delta_{5,5} = 1$

$\because 1 \neq 2$, $\because 5 = 5$

$\because \equiv$ "because"

For an orthonormal set then

$$\langle \alpha_i | \alpha_j \rangle = \delta_{ij}$$

An operator or transformation denoted by a capital letter with a cap on it

$\hat{T}$ is an instruction or recipe to change one vector to another in a precise manner.

$$|\alpha' \rangle = \hat{T} |\alpha\rangle$$

A linear operator is one that has the following property

$$\hat{T}\left(a|\alpha\rangle + b|\beta\rangle\right) = a\left(\hat{T}|\alpha\rangle\right) + b\left(\hat{T}|\beta\rangle\right)$$

$$\forall\ |\alpha\rangle, |\beta\rangle \in S \text{ and } \forall\ a, b \in \mathbb{C},$$

A linear transformation $\hat{T}$ is completely defined by its effect on a basis set $\{ |e_i\rangle,\ i = 1, \ldots n \}$.

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Let $\hat{T}|e_j\rangle = \sum_{i=1}^{n} T_{ij}|e_i\rangle$, $j=1,\ldots n.$

where $T_{ij} \in C$.

Let $|\alpha\rangle = \sum_{j=1}^{n} a_j|e_j\rangle$

Suppose the basis is orthonormal.

Then $\langle e_i|e_j\rangle = \langle e_j|e_i\rangle = \delta_{ij}$

$$\therefore \langle e_i|\hat{T}|e_j\rangle = \langle e_i|\sum_{k=1}^{n} T_{kj}|e_k\rangle$$

$$= \sum_{k=1}^{n} T_{kj}\langle e_i|e_k\rangle = \sum_{k=1}^{n} T_{kj}\delta_{ik}$$

$$= T_{ij}$$

$$\Rightarrow T_{ij} = \langle e_i|\hat{T}|e_j\rangle$$

T_{ij} are called the matrix elements of the operator $\hat{T}$.

Similarly $\langle e_i|\alpha\rangle = \sum_{j=1}^{n} a_j\langle e_i|e_j\rangle$

$$= \sum_{j=1}^{n} a_j\delta_{ij} = a_i$$

$$\Rightarrow a_i = \langle e_i|\alpha\rangle$$

$$a_i^* = \langle\alpha|e_i\rangle$$

a_i are the components or projections of $|\alpha\rangle$ of/on $|e_i\rangle$.

i.e. $a_i \equiv$ component of $|\alpha\rangle$ along $|e_i\rangle$

or projection of $|\alpha\rangle$ on $|e_i\rangle$.

$$\hat{T}|\alpha\rangle = \sum_{j=1}^{n} a_j (\hat{T}|e_j\rangle) = \sum_{j=1}^{n} a_j \left(\sum_{i=1}^{n} T_{ij}|e_i\rangle\right)$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{n} T_{ij}\, a_j\, |e_i\rangle$$

$$= \sum_{i=1}^{n}\left[\sum_{j=1}^{n} T_{ij}\, a_j\right]|e_i\rangle$$

$$= \sum_{i=1}^{n} a_i'\,|e_i\rangle \quad \text{where } a_i' \equiv \sum_{j=1}^{n} T_{ij}\, a_j$$

$\Rightarrow$ $\hat{T}$ takes a vector $|\alpha\rangle$ with components $\{a_i\}$ to a vector $\hat{T}|\alpha\rangle$ with components $\{a_i'\}$.

$$\Rightarrow \begin{bmatrix} a_1' \\ a_2' \\ \vdots \\ a_n' \end{bmatrix} = \begin{bmatrix} T_{11} & \cdots & T_{1n} \\ T_{21} & \cdots & T_{2n} \\ \vdots & & \\ T_{n1} & \cdots & T_{nn} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$\Leftrightarrow \quad a_i' = \langle e_i|\hat{T}|\alpha\rangle \,, \quad \underbrace{\langle e_i|\hat{T}|e_j\rangle}_{=\,T_{ij}}, \quad \underbrace{\langle e_i|\alpha\rangle}_{=\,a_i}.$$

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This means $|\alpha\rangle$ is defined by a column matrix $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}_{n\times 1}$, $\langle\alpha|$ by a

row matrix $\begin{bmatrix} a_1^* & a_2^* & \cdots & a_n^* \end{bmatrix}_{1\times n}$

and the operator $\hat{T}$ by a square matrix

$$\begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & & & \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix}_{n\times n}$$

We will denote such correspondence by $\leftrightarrow$, $\therefore$ $|\alpha\rangle \leftrightarrow \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$.

If $\hat{U} = \hat{S} + \hat{T}$ then

$$U_{ij} = \langle e_i|\hat{U}|e_j\rangle = \langle e_i|\hat{S}+\hat{T}|e_j\rangle$$

$$= \langle e_i|\hat{S}|e_j\rangle + \langle e_i|\hat{T}|e_j\rangle$$

$$= S_{ij} + T_{ij} = U_{ij}$$

Suppose now $\hat{U} = \hat{S}\hat{T}$. This means if $|\alpha'\rangle = \hat{T}|\alpha\rangle$, $|\alpha''\rangle = \hat{S}|\alpha'\rangle$

$$\Rightarrow \quad |\alpha''\rangle = \hat{S}\hat{T}|\alpha\rangle = \hat{U}|\alpha\rangle$$

What is U_{ij} in terms of S_{ij} and T_{ij}.

For this we first note that $\hat{T}$ with matrix elements $\{T_{ij}\}$ can be written as

$$\hat{T} = \sum_{i=1}^{n}\sum_{j=1}^{n} T_{ij}\,|e_i\rangle\langle e_j|$$

We check this now.

$$\therefore \quad \langle e_m|\hat{T}|e_n\rangle = \sum_i \sum_j T_{ij}\,\langle e_m|e_i\rangle\langle e_j|e_n\rangle$$

$$= \sum_i \sum_j T_{ij}\,\delta_{mi}\,\delta_{jn} = T_{mn}$$

as expected.

Now consider the identity operator which leaves all vectors unchanged.

$$\Rightarrow \quad \hat{I}|\alpha\rangle = |\alpha\rangle.$$

~~Let~~ $\Rightarrow \quad \hat{I}|e_j\rangle = |e_j\rangle$

$$\Rightarrow \quad \langle e_i|\hat{I}|e_j\rangle = \langle e_i|e_j\rangle = \delta_{ij}$$

$$\therefore \quad \hat{I} = \sum_i \sum_j \delta_{ij}\,|e_i\rangle\langle e_j| = \sum_i |e_i\rangle\langle e_i|$$

It can be ~~as~~ easily seen that

$$\hat{I}\hat{T} = \hat{T}\hat{I} = \hat{T}, \quad \forall\, \hat{T}.$$

Now $\hat{U} = \hat{S}\hat{T} \Rightarrow U_{ij} = \langle e_i | \hat{S}\hat{T} | e_j \rangle$ A12

Since $\hat{S} = \hat{S}\hat{I}$ we get $\hat{S}\hat{T} = \hat{S}\hat{I}\hat{T}$

$$\therefore \hat{S}\hat{T} = \sum_{k=1}^{n} \hat{S} | e_k \rangle \langle e_k | \hat{T}$$

$$\therefore U_{ij} = \sum_{k=1}^{n} \langle e_i | \hat{S} | e_k \rangle \langle e_k | \hat{T} | e_j \rangle$$

$$= \sum_{k=1}^{n} S_{ik} T_{kj}$$

This is just the formula for product of two matrices. Every operator corresponds to a matrix denoted by $\overline{\overline{T}}$ so that

$$T_{ij} \equiv (\overline{\overline{T}})_{ij} \equiv \langle e_i | \sum_m \sum_n T_{mn} | e_m \rangle \langle e_n | e_j \rangle$$

$$\therefore \overline{\overline{U}} = \overline{\overline{S}}\,\overline{\overline{T}}$$

The Hermitian conjugate of an operator $\hat{T}$ is defined as another operator $\ni$

$$\hat{T}^\dagger \equiv \left[\sum_i \sum_j T_{ij} | e_i \rangle \langle e_j | \right]^\dagger$$

$$\equiv \sum_i \sum_j T_{ij}^* | e_j \rangle \langle e_i |$$

$$\therefore (\hat{T}^\dagger)_{mn} = \langle e_m | \hat{T}^\dagger | e_n \rangle = T_{nm}^*$$

$$\therefore \ (\hat{T}^\dagger)_{mn} = \left[(\hat{T})_{nm}\right]^*$$

If $|\beta\rangle = \hat{T}|\alpha\rangle$ then

$$\langle\beta| = \langle\alpha|\hat{T}^\dagger$$

Also $\quad \overline{\overline{T}}^\dagger \equiv (\overline{\overline{T}}^T)^* \equiv (\overline{\overline{T}}^*)^T$

If $\overline{\overline{T}}$ has elements T_{ij} then the transpose of the matrix $\overline{\overline{T}}$ is defined as

$$\overline{\overline{T}}^T \ni (\overline{\overline{T}}^T)_{ij} = (\overline{\overline{T}})_{ji}$$

i.e. flip all rows with columns.

A unitary operator $\hat{U}$ is defined as one for which

$$\hat{U}^\dagger \hat{U} = \hat{I} = \hat{U}\hat{U}^\dagger$$

A unitary matrix $\ni \ \overline{\overline{U}}^\dagger \overline{\overline{U}} = \overline{\overline{U}}\,\overline{\overline{U}}^\dagger = \overline{\overline{I}}$.

A matrix is called Hermitian iff

$$\overline{\overline{T}}^\dagger = \overline{\overline{T}} \ , \text{ i.e. } T_{ij} = T_{ji}^*.$$

An operator is called Hermitian iff

$$\hat{T}^\dagger = \hat{T}.$$

An anti-Hermitian operator or matrix follow the equations

$$\hat{T}^{\dagger} = -\hat{T} \quad \text{or} \quad \overline{\overline{T}}^{\dagger} = -\overline{\overline{T}}.$$

Consider a vector $|\alpha\rangle$ operated on by some linear operator $\hat{T} \ni$

$$\hat{T}|\alpha\rangle = \lambda|\alpha\rangle, \text{ where } \lambda \in C,$$

is a constant. Such an equation is called an eigen-value equation, the constant λ is called an eigenvalue and $|\alpha\rangle$ is an eigenvector of $\hat{T}$.

If $\hat{T}$ is Hermitian, i.e. $\hat{T} = \hat{T}^{\dagger}$

then we get $\lambda \in R$, i.e. λ is real.

Proof :→ $\hat{T}|\alpha\rangle = \lambda|\alpha\rangle$

$$\Rightarrow \langle\alpha|\hat{T}|\alpha\rangle = \lambda\langle\alpha|\alpha\rangle$$

$$\left[\langle\alpha|\hat{T}|\alpha\rangle\right]^{\dagger} = \langle\alpha|\hat{T}^{\dagger}|\alpha\rangle = \langle\alpha|\hat{T}|\alpha\rangle$$

$$\Rightarrow \left[\langle\alpha|\hat{T}|\alpha\rangle\right]^{\dagger} = \langle\alpha|\hat{T}|\alpha\rangle$$

$$\Rightarrow \lambda^{\dagger}\langle\alpha|\alpha\rangle = \lambda\langle\alpha|\alpha\rangle$$

$$\Rightarrow \lambda^{\dagger} = \lambda \Rightarrow \lambda^{*} = \lambda \Rightarrow \lambda \in R.$$

Typically a linear Hermitian operator has n eigenvalues an n corresponding eigenvectors which are linearly independent. Hence they form a basis in the n dimensional space.

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We write these n equations as

$$\hat{T}|e_i\rangle = \lambda_i |e_i\rangle \quad , \quad i=1,2,\ldots n.$$

$$\therefore \quad (\hat{T} - \lambda_i)|e_i\rangle = |0\rangle$$

$$\Rightarrow \quad (\hat{T} - \lambda_i \hat{I})|e_i\rangle = |0\rangle$$

Since $|e_i\rangle \neq |0\rangle$ the only way to satisfy the eigenvalue equation is if

$$\hat{T} - \lambda_i \hat{I} = \hat{O}$$ where $\hat{O}$ is the

null operator defined $\ni$ $\hat{O}|\alpha\rangle = |0\rangle$

$\forall |\alpha\rangle \in S$. This is only true $\forall$ eigenvectors $|e_i\rangle$ of $\hat{T}$. Since the eigenvectors are linearly independent the condition for solution of such an equation is

$$\det(\bar{\bar{T}} - \lambda_i \bar{\bar{I}}) = 0$$

where $\det(\overline{\overline{M}})$ means the determinant of matrix $\overline{\overline{M}}$.

The inverse of an operator $\hat{T}$ or matrix $\overline{\overline{T}}$ is defined by the equation

$$\hat{T}^{-1}\hat{T} = \hat{T}\hat{T}^{-1} = \hat{I} \quad \text{or}$$

$$\overline{\overline{T}}^{-1}\overline{\overline{T}} = \overline{\overline{T}}\,\overline{\overline{T}}^{-1} = \overline{\overline{I}}.$$

$$\overline{\overline{T}}^{-1} = \frac{\overline{\overline{T}}_c^{\,T}}{\det(\overline{\overline{T}})}$$ where $\overline{\overline{T}}_c$ is defined

as the co-factor matrix of $\overline{\overline{T}}$ and $\overline{\overline{T}}_c^{\,T}$ is the transpose of $\overline{\overline{T}}_c$.