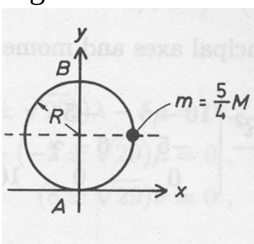
Final Examination for PHYS 6220/7220, Fall 2006

1. A thin disk of radius R and mass M lying the XY plane has a point mass m = (5M)/4 attached to its edge (as shown in Fig. 1). The moment of inertia matrix $\mathbf{I}_D$ of the disk about its center of mass has elements given by $\mathbf{I}_{D\,i,j} = \left(\frac{MR^2\delta_{i,j}}{4}\right)(1+\delta_{i,3})$ where $\delta_{i,j}$ is the Kronecker delta function. The coordinate system used to calculate $\mathbf{I}_D$ is parallel to the one shown in the Fig. 1.

Fig. 1



(a) Find the moment of inertia matrix $\mathbf{I}_{DA}$ of the disk about point A in the coordinate system shown in Fig. 1. (**2 point**)
(b) Find the moment of inertia matrix $\mathbf{I}_{PA}$ of the point mass about point A in the coordinate system shown Fig. 1. (**2 point**)
(c) Find the total moment of inertia matrix $\mathbf{I}_{TA}$ about point A in the coordinate system shown in Fig. 1. (**1 point**)
(d) Write the appropriate equation D = 0, where D is the determinant containing the eigenvalue λ of $\mathbf{I}_{TA}$. (**1 point**)
(e) By a careful observation of the result in (d) find the first value of λ. (**1 point**)
(f) Then using result in (e) find the other two values. Keep irrational numbers in their square root form. Do not convert any answers to decimals. (**2 points**)
(g) Find the eigenvector of the matrix $\mathbf{I}_{TA}$ corresponding to the first eigenvalue found in part (e). (**1 points**)

2. A one dimensional rod is constrained to lie in the XY plane. Its end points are denoted by points P and Q. Point P which is closer to the origin maintains a fixed distance b from the origin O. Point P moves continuously in a circle with fixed angular frequency ω anti-clockwise around the Z axis. An ant of mass m moves from point P to point Q on the rod, with constant speed v_0 with respect to the rod. It is located at point P when it starts its motion. The rod lies along the X axis when the ant starts its motion. Express all answers in terms of b, v_0, ω and unit vectors along the coordinate axes. Parts (a) and (b) describe two independent motions.
(a) If the points P, Q, and O are constrained to be collinear at all times then compute the total force on the ant, caused by the non-intertial motion of the rod, as a function of time. (**2 points**)
(b) If the points P and Q are constrained to lie parallel to the X axis at all times compute the total force on the ant, caused by the non-intertial motion of the rod, as a function of time. (**2 points**)

3. A particle of mass m moves under gravity on a smooth surface the equation of which is $z = x^2 + y^2 - xy$. The Z axis is taken along the vertical pointing upwards. The magnitude of the acceleration due to gravity is g.
(a) Write an expression for the potential energy of the particle purely as a function of x and y, V = V(x, y). (**1 point**)
(b) Find the point (x_0, y_0) at which the potential is an extremum. (**1 point**)
(c) Compute the kinetic energy $T = T(x, y, \dot{x}, \dot{y})$. (**1 point**)

(d) Construct the appropriate Lagrangian for small oscillations about the point (x_0, y_0). Find the eigenfrequencies for these oscillations. (**2 points**)
(e) Find the corresponding eigenvectors. (**2 points**)
(f) Find the most general solution. (**1 point**)

4. A particle of mass m and magnitude of initial angular momemtum ℓ moves in a central force field such that r = kθ, where r is its distance from the center of force and the origin is taken at the center of force. The angle its position vector makes with the positive X axis is defined to be θ. Express all answers in terms of m, k, and ℓ only.
(a) Find the form of the central potential V = V(r). (**2 points**)
(b) Find r = r(t). (**1 point**)

5. A particle of mass m is constrained to move on a sphere of radius a about the origin O. It is attracted by a force given by $\overline{F} = -k\left(x\hat{i} + y\hat{j}\right)$, where (x, y, z) are the Cartesian coordinates of the particle and $\hat{i}$, $\hat{j}$ are unit vectors along X and Y axis. Express all answers in terms of a, m, and k only.
(a) Define generalized coordinates from a cylindrical coordinate system and hence write the Lagrangian for the particle. (**2 points**)
(b) Write expressions for all constants of the motion in terms of generalized coordinates. (**2 points**)
(c) Use these constants to show that the problem may be transformed into an effective one dimensional problem. (**1 point**)

Some relevant and irrelevant information.

A] $\sqrt{116} = 2\left(\sqrt{29}\right)$.