Examination I for PHYS 6220/7220, Fall 2005

1. In an unknown universe there are new laws of mechanics. The Lagrangian for a single coordinate is given by $L = L(q, \dot{q}, \ddot{q}, t)$, instead of the usual $L = L(q, \dot{q}, t)$. The Lagrange equations of motion for this Lagrangian are given by

$$\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \left(\frac{\partial L}{\partial q}\right) = 0 \tag{1}$$

(a) Apply this result to the particular Lagrangian given by

$$L = -\left(\frac{mq\ddot{q}}{2} + \frac{bq^2}{2}\right), \tag{2}$$

to obtain the equation of motion. The mass of the particle is m and b is a constant of appropriate dimensions. (2 points)

(b) Write the most general solution to the equation of motion. You need not solve it step by step. (1 point).

(c) Write the Lagrangian for the problem in our regular universe which would lead to the same type of motion (1 point).

2. A particle of of mass m is suspended by a massless string of length L. It hangs, without initial motion, in a gravitational field of strength g. It is struck by an impulsive horizontal blow, which introduces an angular velocity ω at t = 0. If ω is sufficiently small, it is clear that the mass moves as a simple pendulum. If ω is sufficiently large, the mass will rotate about the support. Assume there is no friction or air resistance.

(a) Use an appropriate set of coordinates to obtain the Lagrangian of the system. (2 points)

(b) Use a Lagrange multiplier λ associated with the constraint that the string has a fixed length L to write down the Lagrange equations of motion. (2 point)

(c) Solve for λ in terms of the angle θ the string makes with a vertical line and other given constants. (1 point)

(d) What are the constants of motion? Express them in terms of the generalized velocities and coordinates. (1 point).

(e) If at a certain time $\lambda = 0$ becomes true what will happen to the motion of the particle?

(1 point)

(f) What is the value of the angle θ when this happens? (1 point)

(g) If the particle has to complete one full rotation what should be the minimum value of ω? (1 point)

3. A Lagrangian of coordinate q is given by

$$L = \frac{m}{2}\left(\dot{q}^2 \sin^2(\omega t) + \omega q \dot{q} \sin(2\omega t) + q^2 \omega^2 \cos^2(\omega t)\right) - \frac{kq^4}{2} \sin^4(\omega t), \tag{3}$$

where m, k, and ω are constants.

(a) Is the corresponding Hamiltonian H(q,p,t) conserved? Give a reason for your answer. You need not obtain H. (1 point)

(b) Make a substitution $s = q\sin(\omega t)$, to evaluate $\dot{s}$ and hence obtain $L(s, \dot{s}, t)$. (2 points)

(c) Find the new Hamiltonian, $H(s, p_s, t)$. (3 points)

(d) Is the corresponding Hamiltonian conserved? Give a reason for your answer. (1 point)

Some relevant and irrelevant formulae

(1) $\sin(2x) = 2\sin(x)\cos(x)$ and $\cos(2x) = \cos^2(x) - \sin^2(x)$.

(2) If x = x(t) then $\ddot{x} = \dot{x}\left(\frac{d\dot{x}}{dx}\right)$.