For a central force problem we have
\[ F(\vec{r}) = f(\vec{r}) \frac{\vec{r}}{r} \]

\[ \therefore \quad \frac{d}{dt} \vec{p} = f(\vec{r}) \frac{\vec{r}}{r} \]

\[ \vec{I} \equiv \vec{r} \times \vec{p} = \text{constant} \]

\[ \therefore \quad \frac{d}{dt} (\vec{r} \times \vec{I}) = \vec{p} \times \vec{I} = f(\vec{r}) \frac{\vec{r}}{r} \times \vec{I} \]

\[ = m \frac{f(\vec{r})}{r} \left[ \vec{r} \times (\vec{r} \times \vec{r}) \right] \]

\[ = m \frac{f(\vec{r})}{r} \left[ \vec{r} (\vec{r} \cdot \vec{r}) - (\vec{r} \cdot \vec{r}) \vec{r} \right] \]

where we used \( \vec{A} \times (\vec{B} \times \vec{C}) = \vec{B} (\vec{A} \cdot \vec{C}) - \vec{C} (\vec{A} \cdot \vec{B}) \)

\[ \therefore \quad \frac{d}{dt} (\vec{r} \times \vec{I}) = m \frac{f(\vec{r})}{r} \left[ \vec{r} \frac{d}{dt} (\vec{r} \cdot \vec{r}) - \vec{r} \frac{d}{dt} \vec{r} \right] \]

\[ = m \frac{f(\vec{r})}{r} \left[ \vec{r} \vec{r} \vec{r} - \vec{r} \vec{r} \vec{r} \right] \]

\[ = m \frac{f(\vec{r})}{r} \left[ \vec{r} \vec{r} \vec{r} - \vec{r} \vec{r} \vec{r} \right] = m \frac{f(\vec{r})}{r^2} \left[ \vec{r} \vec{r} \vec{r} - \vec{r} \vec{r} \vec{r} \right] \]

\[ = -m \frac{f(\vec{r})}{r^2} \left[ \vec{r} \vec{r} \vec{r} - \vec{r} \vec{r} \vec{r} \right] = -m \frac{f(\vec{r})}{r^2} \frac{d}{dt} (\vec{r}) \]

\[ \therefore \quad \frac{d}{dt} (\vec{r} \times \vec{I}) + m \frac{f(\vec{r})}{r^2} \frac{d}{dt} (\vec{r}) = 0 \]
If \( V(r) = -\frac{k}{r} \) then \( f(r) = -\frac{\partial V}{\partial r} = -\frac{k}{r^2} \)

\[
\therefore \frac{d}{dt} \left[ \frac{\mathbf{p} \times \mathbf{L} - mk\mathbf{r}}{r} \right] = 0
\]

\( \mathbf{p} \times \mathbf{L} - mk\mathbf{r} = \mathbf{A} = \text{constant vector} \)

\( \mathbf{A} \) is called the Laplace–Runge–Lenz vector, since \( \mathbf{A} \) is constant let us calculate it at the perihelion.

\[ r_c = \frac{a(1-e^2)}{1+e\cos\theta} \]

\[ r_{min} = a(1-e) \]

\( \mathbf{p} = m (\dot{r} \mathbf{\hat{r}} + r \dot{\theta} \mathbf{\hat{\theta}}) \).

at \( r = r_{min}, \dot{r} = 0 \)

\( \Rightarrow \mathbf{p} = mr \dot{\theta} \mathbf{\hat{\theta}} \)

\( \mathbf{L} = l \mathbf{\hat{z}} = \text{normal to the plane} \)

\( \Rightarrow \mathbf{p} \times \mathbf{L} = m r \dot{\theta} (\mathbf{\hat{\theta}} \times \mathbf{\hat{z}}) = m r_{min} \dot{\theta} \mathbf{\hat{r}} \)

\( \therefore |\mathbf{A}| = |\mathbf{p} \times \mathbf{L} - mk\mathbf{\hat{r}}| = m [l r_{min} \dot{\theta} - k] \)
But \( \dot{\theta} = \frac{d}{(m \tau^2)} \)

\[ l \tau_{\text{min}} \dot{\theta} = \frac{l^2}{(m \tau_{\text{min}})} = \frac{l^2}{ma(1-\epsilon)} \]

Also we have

\[ \frac{l^2}{mk} = a(1-\epsilon^2) \]

\[ \therefore \frac{l^2}{ma(1-\epsilon)} = k[1+\epsilon] \]

\[ \therefore |A| = mk \epsilon \]

Also \( \epsilon = \sqrt{1 + \frac{2El^2}{mk^2}} \)

\[ \therefore |A|^2 = m^2k^2 + 2mEl^2 \quad 3.87 \]

Note also that there are constraints we have discovered \( \mathbf{A}, \mathbf{I}, \mathbf{E} \) since \( \mathbf{A} \neq \mathbf{L} \)

are vectors,

\[ I \otimes (\mathbf{p} \times \mathbf{L}) = 0 \] also \( I \cdot \mathbf{r} = (\mathbf{r} \times \mathbf{p}) \cdot \mathbf{r} = 0 \)

\[ \therefore I \cdot \mathbf{A} = 0 \quad \rightarrow 3.83 \]

With the constraints \( 3.83 \) and \( 3.87 \) there are only 5 independent constants.
FIGURE 3.18 The vectors $p$, $L$, and $A$ at three positions in a Keplerian orbit. At perihelion (extreme left) $|p \times L| = mk(1+e)$ and at aphelion (extreme right) $|p \times L| = mk(1-e)$. The vector $A$ always points in the same direction with a magnitude $mk$.

the orbit. If $\theta$ is used to denote the angle between $r$ and the fixed direction of $A$, then the dot product of $r$ and $A$ is given by

$$A \cdot r = Ar \cos \theta = r \cdot (p \times L) - mkr.$$  \hspace{1cm} (3.84)

Now, by permutation of the terms in the triple dot product, we have

$$r \cdot (p \times L) = L \cdot (r \times p) = I^2,$$

so that Eq. (3.84) becomes

$$Ar \cos \theta = I^2 - mkr,$$

or

$$\frac{1}{r} = \frac{mk}{I^2} \left(1 + \frac{A}{mk} \cos \theta \right).$$ \hspace{1cm} (3.85)

The Laplace–Runge–Lenz vector thus provides still another way of deriving the orbit equation for the Kepler problem! Comparing Eq. (3.85) with the orbit equation in the form of Eq. (3.55) shows that $A$ is in the direction of the radius vector to the perihelion point on the orbit, and has a magnitude

$$A = mke.$$ \hspace{1cm} (3.86)

For the Kepler problem we have thus identified two vector constants of the motion $L$ and $A$, and a scalar $E$. Since a vector must have all three independent components, this corresponds to seven conserved quantities in all. Now, a system such as this with three degrees of freedom has six independent constants of motion, corresponding, say to the three components of both the initial position and velocity vectors. Consequently, to avoid infinite solutions, we must require a further condition. For the central force problem under consideration, this is the requirement that the orbital energy be negative. If, however, the orbital energy is positive, the position and velocity vectors do not exist but...
Scattering in a central force

The differential cross section

\[ d\sigma_R(\vec{n}) = \sigma_R(\vec{n}) \, d\Omega = \frac{N_R(\vec{n})}{I} \]

where \( I = \) incident intensity

\[ N_R(\vec{n}) = \] number of particles scattered into the solid angle \( d\Omega \) per unit time,

\[ [I] = [M^0 L^{-2} T^{-1}] \]

\[ [N_R(\vec{n})] = [M^0 L^0 T^{-1}] \]

\[ [d\sigma_R(\vec{n})] = [M^0 L^2 T^{-1}] \]

In central force problems, \( I \) symmetry around the axis of the incident beam which is assumed to pass through the center of force.

\[ \therefore \quad d\Omega = 2\pi \sin \Theta_R \, d\Theta_R \]

where \( \Theta_R = \) angle between the scattered and incident directions.
Let $v_0 =$ initial speed of the incident particles, and $s =$ impact parameter.

$s =$ distance from the center of force of the line determined by the initial particle position and velocity.

\[ |I| = l = \frac{mv_0}{s} = s\sqrt{2mE} \rightarrow 3.90 \]

since $2E = \frac{mv_0^2}{2} =$ initial energy $\times 2$.

Assume for simplicity that $s$ determines $\theta_R$ uniquely.

\[ 2\pi l s|ds| = 2\pi \sigma_R(\theta_R) I \sin \theta_R |d\theta_R| \]

where we denote $\sigma_R(\theta_R) = \sigma_R(\theta)$ for central forces.

\[ \sigma_R(\theta_R) = \frac{s}{\sin(\theta_R)} |\frac{ds}{d\theta_R}| \rightarrow 3.93 \]
From figure we see that

\[ \Theta_R + 2 \Psi = \Pi \quad \text{where } \Psi \text{ is the} \]

angle between the incident velocity \( \vec{V}_0 \) and the line from the center of force to the point of closest approach. The latter is called the periapsis.

Eq. (3.36) reads

\[ \Theta = \int \frac{r}{r_0 \sqrt{2mE \xi^4 - 2mV_r \xi^4 - \xi^2}} \, dr + \Theta_0 \]

At the initial point \( r_0 = \infty \) and \( \Theta_0 = 0 \Pi \)

Also \( \Theta \odot \Psi = \Theta_R \rightarrow (3.945) \)

\[ (3.945) \text{ with (3.94) } \Rightarrow \Psi = \Pi - \Theta, \]

which with (3.36) gives

\[ \Psi = \int \frac{1}{r_0 \sqrt{2mE \xi^4 - 2mV_r \xi^4 - \xi^2}} \, dr \]

\[ \therefore \Theta_R = \Pi - 2 \Psi = \Pi - 2 \int_{r_m}^{\infty} \frac{s \, ds}{r_m \left[ \xi^4 \left( 1 - \frac{V(r)}{E} \right) - s^2 \xi^2 \right]^{1/2}} \]
Let \( u = 1/r \)

\[
\Theta_R(s) = \pi - 2 \int_0^u \frac{sdu}{\sqrt{1 - \frac{V(1/u) - s^2u^2}{E}}}
\]

Consider now the center of force to have a charge \(-Ze\), \(e = \) proton charge. Let the scattering particle have charge \(-Z'e\), \(Z' > 0\), \(Z' > 0\)

- \( \text{force } f = \frac{ZZ'e^2}{r^2} \)

- \( V(r) = \frac{ZZ'e^2}{r} = -k/r, \quad k = ZZ'e^2 \)

Note \( k < 0 \) \( \Rightarrow \) repulsive potential.

\[
E = \frac{mv_0^2}{2} - 70 \Rightarrow E > 71
\]

By (3.57) \( E = \sqrt{1 + \frac{2E}{mk^2}} \)

which using (3.98) and (3.90) becomes

\[
E = \sqrt{1 + \left( \frac{2Es}{ZZ'e^2} \right)^2}
\]

\[ \Rightarrow (3.99) \]
Eq. 3.55 gives
\[ \rho = \frac{2}{mk[1 + E \cos(\theta - \theta_0)]} \]

\[ \rho \to 0 \Rightarrow \cos(\theta - \theta_0) = -1/\epsilon \]

Let \( \theta - \theta_0 = \pi - \gamma \)

\[ \Rightarrow \cos \gamma = 1/\epsilon \]

\[ \rho_R = \pi - 2\gamma \Rightarrow \gamma = \left( \frac{\pi - \theta_R}{2} \right) \]

\[ \Rightarrow \cos \left( \frac{\pi - \theta_R}{2} \right) = \sin \left( \frac{\theta_R}{2} \right) = 1/\epsilon \]

\[ \Rightarrow \cot^2 \left( \frac{\theta_R}{2} \right) = \csc^2 \left( \frac{\theta_R}{2} \right) - 1 = \epsilon^2 - 1 \]

\[ = \frac{2Es}{ZZ'e^2} \]

\[ \Rightarrow s = \frac{ZZ'e^2 \cot(\theta_R)}{2E} \]

By 3.93
\[ \sigma_R(\theta_R) = 1 \left( \frac{ZZ'e^2}{2E} \right)^2 \csc^4 \left( \frac{\theta_R}{2} \right) \]

The total cross-section:
\[ \sigma_{RT} = \int \sigma_R(\theta) \, d\Omega = 2\pi \int_0^{\pi} \sigma_R(\theta_R) \sin(\theta_R) \, d\theta_R \]

\[ = \int_0^{\pi} \sigma_R(\theta_R) \sin(\theta_R) \, d\theta_R \]
$\sigma_{RT} \to \infty$ for Coulomb scattering.

Scattering in the laboratory frame:

Let $\mathbf{v}_0$ = initial velocity of $m_1$.
Assume $m_2$ is at rest initially.
Let $\mathbf{v}_i$ = final velocity of $m_1$.

\[ m_1 \mathbf{v}_0 = (m_1 + m_2) \mathbf{v} \]

\[ \mathbf{v} \equiv \text{COM velocity} \]

From figure above:

\[ v_i \sin \theta_L = v_i' \sin \theta_R \]

\[ v_i \cos \theta_L = v_i' \cos \theta_R + v \]

\[ \tan \theta_L = \frac{\sin \theta_R}{\cos \theta_R + \rho} \]

\[ \rho = \frac{v_i'}{v} = \frac{v_i'(m_1 + m_2)}{v_0 m_1} \]

\[ \cos \theta_L = \left(1 + \tan^2 \theta_L\right)^{-1/2} = \frac{\rho + \cos \theta_R}{\sqrt{1 + \rho^2 + 2 \rho \cos \theta_R}} \]
Also
\[ \overline{v}_1' = \overline{v}_1 - \overline{v} = \overline{v}_1 - \frac{m_1 \overline{v}_1 + m_2 \overline{v}_2}{m_1 + m_2} \]
\[ = \frac{m_2 \overline{v}_{12}}{m_1 + m_2} \]
where \( \overline{v}_{12} = \overline{v}_1 - \overline{v}_2 \)

and \( \overline{v}_2 = \) velocity of the second particle after collision.

\[ ' , | \overline{v}_1' | = \overline{v}_1' = \frac{m_2 \overline{v}_{12}}{m_1 + m_2} \]

\[ ' , \rho = \frac{m_1 v_0}{m_2 \overline{v}_{12}} \]

Let \( \phi \) of the collision be defined by

\[ \phi = \frac{1}{2} \left( \frac{m_1 m_2}{m_1 + m_2} \right) \left( \frac{\overline{v}_{12}^2 - v_0^2}{(m_1 + m_2)} \right) \]
\[ = \frac{1}{2} \left( \frac{m_1 v_0^2}{m_1 + m_2} \right) \left( \frac{(m_1 + m_2)}{v_0^2} \right) \]

\[ \therefore \overline{v}_{12} = \sqrt{1 + \frac{(m_1 + m_2) \phi}{m_2 E}} \]

\[ \therefore \rho = \frac{m_1 E'^{1/2}}{\sqrt{m_2 (E + \phi) + m_1 m_2 \phi}} \]

For elastic scattering there is no energy loss \( \Rightarrow \phi = 0 \)
\[ \rho = \frac{m_1}{m_2} \rightarrow 3.111 \]

Let \( \sigma_\perp(\theta_L) = \sigma_\perp(\theta_L)|d\theta_L|\ 2\pi \sin(\theta_L) \)

be the differential cross-section as measured in the laboratory, with \( \theta_L \) as argument.

Conservation of particles

\[ 2\pi I \sigma_\perp \sin(\theta_R)|d\theta_R| = 2\pi I \sigma_\perp(\theta_L) \sin \theta_L |d\theta_L| \]

\[ \sigma_\perp(\theta_L) = \sigma(\theta_R) \left| \frac{d(\cos \theta_R)}{d(\cos \theta_L)} \right| \rightarrow 3.115 \]

Using (3.111) in (3.115) we get

\[ \sigma_\perp(\theta_L) = \sigma(\theta_R) \left[ 1 + 2 \rho \cos \theta_R + \rho^2 \right] \frac{3}{2} \]

\[ 1 + \rho \cos \theta_R \]

Let \( E_1 = \frac{m_1 v_i^2}{2} = \text{final energy of the incident particle} \)

\[ \therefore E, / E = \frac{v_i^2}{V_o^2} = \left( \frac{v_i + \bar{V}}{V_o} \right)^2 = \frac{v_i^2 + V^2 + 2v_i' V \cos \theta_R}{V_o^2} \]

\[ = \frac{v_i'^2}{V_o^2} + \frac{V^2}{V_o^2} + \frac{2v_i' \cos \theta_R}{V_o} = \left( \frac{m_i}{m_i + m_2} \right) \left[ \frac{1 + \rho^2 + 2 \rho \cos \theta_R}{\rho^2} \right] \]