$$\Rightarrow \quad \langle T \rangle + \frac{1}{2} \langle \sum_i \bar{F}_i \cdot \bar{r}_i \rangle = 0$$

$$\Rightarrow \quad \langle T \rangle = -\frac{1}{2} \langle \sum_i \bar{F}_i \cdot \bar{r}_i \rangle$$

where $\langle T \rangle$ denote a long time average of T and similarly for $\langle \bar{F}_i \cdot \bar{r}_i \rangle$.

If $\quad V(r) = a r^{n+1}$

$\Rightarrow$ for a single particle

$$F(r) = -a(n+1) r^n$$

$$r F(r) = -(n+1) V(r)$$

$$\Rightarrow \quad \langle T \rangle = \left(\frac{n+1}{2}\right) \langle V(r) \rangle$$

Conditions for closed orbits : →

For a circular ~~or~~ orbit we need $\dot{r} = 0$

$$\Rightarrow \quad E - V_e(r) \Big|_{r=r_0} = \text{an extremum}$$

$$\Rightarrow \quad \frac{dV_e(r)}{dr}\Big|_{r=r_0} = 0$$

For stable periodic orbits we need

$$\frac{d^2 V_e}{dr^2}\Big|_{r=r_0} > 0$$

$\boxed{3\text{-}14}$

$$V_e = V(r) + \frac{l^2}{2mr^2}$$

$$\left.\frac{dV_e}{dr}\right|_{r=r_0} = \left.\frac{dV(r)}{dr}\right|_{r=r_0} - \frac{l^2}{mr_0^{2+1}} = 0$$

$$\Rightarrow \quad \frac{l^2}{mr_0^{2+1}} = \left.\frac{dV}{dr}\right|_{r=r_0} = \frac{l^2}{mr_0^3}$$

$$\frac{d^2V_e}{dr^2} = \left.\frac{d^2V}{dr^2}\right|_{r_0} + \frac{3l^2}{mr_0^4} > 0$$

$$\Rightarrow \quad \left.\frac{d^2V}{dr^2}\right|_{r_0} + \frac{3}{r_0}\left.\frac{dV}{dr}\right|_{r_0} > 0$$

If $V(r) = +\frac{kr^{n+1}}{(n+1)}$

$$\Rightarrow \quad [dV/dr]_{r_0} = +k\frac{(n+1)}{n+1} r_0^n$$

$$\left.\frac{d^2V}{dr^2}\right|_{r_0} = +k\frac{(n+1)}{n+1} n r_0^{n-1}$$

$\Rightarrow$ for stability we need

$$+(n+1)n + 3(n+1) > 0$$

$$+n + 3 > 0 \quad \Rightarrow -(n+3) < 0$$

$$n > -3$$

Bertrand's Theorem: $\rightarrow$ If $V(r) = \underline{kr^{n+1}}$ then closed orbits are stable only for $n=1$ or $n=-2$.

Equation for the orbit :- →

Differential form

$$L = T - V(r)$$

$$= \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$\Rightarrow \frac{d}{dt}(m\dot{r}) + \frac{\partial V(r)}{\partial r} - mr\dot{\theta}^2 = 0$$

$$\dot{\theta} = l/(mr^2)$$

$$\Rightarrow m\ddot{r} + \frac{\partial V(r)}{\partial r} - \frac{l^2}{mr^3} = 0$$

Now $\dot{\theta} = \frac{d\theta}{dt} = \frac{l}{mr^2}$

$$\Rightarrow dt = \frac{mr^2}{l}\,d\theta$$

$$\therefore \frac{d}{dt}\left(\frac{d}{dt}(r)\right) = \frac{l^2}{m^2r^2}\frac{d}{d\theta}\left(\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)\right)$$

$$u = 1/r \Rightarrow r = 1/u, \quad \frac{d(1/u)}{d\theta} = \frac{-1}{u^2}\frac{du}{d\theta}$$

$$\therefore \ddot{r} = \frac{-l^2 u^2}{m^2} \frac{d^2 u}{d\theta^2}$$

$$\therefore \frac{-l^2 u^2}{m} \frac{d^2 u}{d\theta^2} - \frac{l^2 u^3}{m} = +u^2 \frac{d}{du}\left(V(1/u)\right)$$

$$\therefore \frac{d^2 u}{d\theta^2} + u = -\frac{m}{l^2} \frac{d}{du}\left[V(1/u)\right] \rightarrow (3.34)$$

Integral form : $\rightarrow$

We have $\dot{\theta} = l/(mr^2)$

$$\dot{r} = \left[\frac{2}{m}\left[E - V(r) - \frac{l^2}{2mr^2}\right]\right]^{1/2}$$

$$\therefore \frac{dr}{d\theta} = \dot{r}/\dot{\theta} = \frac{\dot{r} m r^2}{l}$$

$$\therefore \int_{\theta_0}^{\theta} d\theta = \theta - \theta_0 = \int_{r_0}^{r} \frac{l\, dr}{r^2\left[(2m)\left[E - V(r) - \frac{l^2}{2mr^2}\right]\right]^{1/2}}$$

$$\therefore \theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - u^2}}$$

If we let $V(r) = ar^{n+1}$

$V(1/u) = au^{-n-1}$

$$\therefore \quad \theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2m}{l^2}\left(E - a u^{-n-1}\right) - u^2}} \rightarrow \text{(3.39)}$$

$n = 1, -2, -3$ yield trigonometric functions.

3-17

$n = 5, 3, 0, -4, -5, -7$ yield elliptic integrals.

The Kepler problem :→

For this problem $n = -2$, $a = -k$

$$\therefore \quad \theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{l^2} + \frac{2mku}{l^2} - u^2}}$$

which ~~evalut~~ evaluates to

$$\theta = \theta_0 - \arccos\left[\frac{\left(\frac{l^2 u}{mk} - 1\right)}{\sqrt{1 + \frac{2El^2}{mk^2}}}\right]$$

$$\therefore \quad \frac{1}{r} = \frac{mk}{l^2}\left(1 + e\cos(\theta - \theta_0)\right)$$

where $e \equiv \sqrt{1 + \frac{2El^2}{mk^2}} \equiv \epsilon$

Another simpler solution is to use ~~3.33~~ (3.34) with

$$V(r) = -k/r \Rightarrow V(1/u) = -ku$$

$$\Rightarrow \frac{d}{du} V(1/u) = -k$$ 3-18

$$\therefore \quad \frac{d^2u}{d\theta^2} + u = \frac{mk}{l^2}$$

Let $z = u - \frac{mk}{l^2}$

$$\Rightarrow \frac{d^2 z}{d\theta^2} + z = 0$$

$$\Rightarrow \quad z = B\cos(\theta - \theta_0) \text{ where}$$

B and θ_0 are integration constants

$$\therefore \quad u = \frac{mk}{l^2}\left(1 + \epsilon\cos(\theta - \theta_0)\right)$$

where ϵ is to be determined

$$\frac{mk}{l^2}\epsilon \equiv B$$

$$\therefore \quad \frac{1}{r} = \frac{mk}{l^2}\left(1 + \epsilon\cos(\theta - \theta_0)\right)$$

The general equation for a conic section is

$$1/r = C\left[1 + \epsilon\cos(\theta - \theta_0)\right]$$

where ϵ = eccentricity

$$\epsilon = \sqrt{1 + \frac{2El^2}{mk^2}}$$

$\epsilon > 1 \Leftrightarrow E > 0 \Leftrightarrow$ hyperbola

$\epsilon = 1 \Leftrightarrow E = 0 \Leftrightarrow$ parabola

$\epsilon < 1 \Leftrightarrow E < 0 \Leftrightarrow$ ellipse

$\epsilon = 0 \Leftrightarrow E = -\frac{mk^2}{2l^2} \Leftrightarrow$ circle

Consider $\epsilon < 1$ case.

$$\Rightarrow r = \frac{l^2}{mk(1 + \epsilon\cos(\theta - \theta_0))}$$

$r = r_{min}$ when $\cos(\theta - \theta_0) = +1$

$r = r_{max}$ when $\cos(\theta - \theta_0) = -1$

$$r_{max} = \frac{l^2}{mk(1-\epsilon)}, \quad r_{min} = \frac{l^2}{mk(1+\epsilon)}$$

The semimajor axis $a \equiv \frac{r_{min} + r_{max}}{2}$

$$= \frac{l^2}{mk}\left(\frac{1}{1-\epsilon^2}\right) = \frac{l^2}{mk} \frac{1}{\frac{-2El^2}{mk^2}} = \frac{-k}{2E}$$

$$\therefore\ a = \frac{-k}{2E} \Rightarrow E = \frac{-k}{2a}$$

$$\Rightarrow\ \epsilon = \sqrt{1 + \left(\frac{-k}{a}\right)\frac{l^2}{mk^2}} = \sqrt{1 - \frac{l^2}{mka}}$$

$$\therefore\ l^2/(mk) = a(1-\epsilon^2)$$

$$\Rightarrow\ r = \frac{a(1-\epsilon^2)}{1+\epsilon\cos(\theta-\theta_0)}$$

For $\epsilon = 0$ we have ~~from~~ $r = $ const.

$\Rightarrow \dot{r} = 0$, $\dot{\theta} = \dfrac{l}{mr^2} = $ const.

$T = $ const., $V = -k/r = $ const.

By Virial Theorem for $V = -k/r$ we have for a circular orbit

$$\langle T \rangle = T = +\frac{(-2+1)}{2}\langle V \rangle = \frac{-V}{2}$$

$$E = T + V = V/2$$

$$\therefore\ E = \frac{-k}{2r_0}.$$

Motion in time: $\rightarrow$

What is $r = r(t)$, $\theta = \theta(t)$.

We have

$$t = \int_{r_0}^{r} dr \left[\frac{2}{m} \left(E - V(r) - \frac{\ell^2}{2mr^2} \right) \right]^{-1/2}$$

$$= \left(\frac{m}{2}\right)^{1/2} \int_{r_0}^{r} \frac{dr}{\sqrt{E + \frac{k}{r} - \frac{\ell^2}{2mr^2}}}$$

We also have $2a = -\frac{k}{E}$

and $\frac{\ell^2}{mk} = a(1-\epsilon^2)$

$$\therefore t = \left(\frac{m}{2k}\right)^{1/2} \int_{r_0}^{r} \frac{r\,dr}{\sqrt{r - \frac{r^2}{2a} - \frac{a(1-\epsilon^2)}{2}}}$$

Choose r_0 as the perihelion distance and make the substitution

$$r = a(1 - \epsilon\cos\Psi) \;,\; \Psi = 0 \;,\; \forall r = r_0$$

$$\therefore t = \left(\frac{ma^3}{k}\right)^{1/2} \left[\Psi - \epsilon \sin\Psi\right]$$

Since $\Psi = \Psi(r)$ we have in principle solved for $r = r(t)$ through $t = t(\Psi)$. Ψ is called the eccentric anamoly. Similarly we have

$$r = a(1-\epsilon\cos\Psi) = \frac{a(1-\epsilon^2)}{1+\epsilon\cos(\theta-\theta_0)}$$

[Note we chose $r_0 = a(1-\epsilon)$
which is naturally satisfied
at $\theta = \theta_0$.]

$\therefore$ we get $\cos(\theta-\theta_0) = \dfrac{\cos\Psi - \epsilon}{1-\epsilon\cos\Psi}$

which may be rewritten as

$$\tan\left(\frac{\theta-\theta_0}{2}\right) = \left(\frac{1+\epsilon}{1-\epsilon}\right)^{1/2}\tan\left(\frac{\Psi}{2}\right).$$

So in principle we also have
$t = t(\Psi(\theta)) = t(\theta) \Rightarrow \theta = \theta(t)$.

Our solution does work $\forall \epsilon \neq 1$
As $\epsilon \to 1$, $a \to \infty$ for r to be finite
and non-zero.
So $\forall \epsilon = 1$ we have $E = 0$

We use $\therefore$ $mr^2\dot{\theta} = l$
$$\Rightarrow \quad d\theta = \frac{l}{mr^2}\,dt$$

$$\therefore \int dt = \frac{m}{l}\int r^2\,d\theta$$

$$\therefore t = \frac{l^3}{mk^2}\int_{\theta_0}^{\theta} d\theta\,\left[1+\epsilon\cos(\theta-\theta_0)\right]^{-2}$$

This is true $\forall \epsilon$ in $[0,\infty]$

Now consider $\epsilon = 1$

Then $$t = \frac{\ell^3}{mk^2}\int_{\theta_0}^{\theta} d\theta \left[1+\cos(\theta-\theta_0)\right]^{-2}$$

$$1+\cos(\theta-\theta_0) = 2\cos^2\left(\frac{\theta-\theta_0}{2}\right)$$

$$\therefore t = \frac{\ell^3}{4mk^2}\int_{\theta_0}^{\theta} d\theta \left[\sec^4\left(\frac{\theta}{2}\right)\right]$$

$$= \frac{\ell^3}{4mk^2}\int_{\theta_0}^{\theta} d\theta \left[\sec^2\left(\frac{\theta-\theta_0}{2}\right)\left[1+\tan^2\left(\frac{\theta-\theta_0}{2}\right)\right]\right]$$

$$= \frac{2\ell^3}{4mk^2}\int_{0}^{\tan\left(\frac{\theta-\theta_0}{2}\right)} (1+x^2)\,dx$$

$$t = \frac{\ell^3}{2mk^2}\left[\tan\left(\frac{\theta-\theta_0}{2}\right)\right]\left[1+\frac{1}{3}\tan^2\left(\frac{\theta-\theta_0}{2}\right)\right]$$

Here $-\pi < \theta < \pi$.

As $t \to -\infty$, $\theta \to -\pi$

$t \to 0$, $\theta \to 0$

$t \to \infty$, $\theta \to \pi$,

Now let's get back to the ellipse $0 < \epsilon < 1$.

$$t = \sqrt{\frac{ma^3}{k}} \int_0^{\Psi} \left[1 - \epsilon \cos \Psi\right] d\Psi$$ $\boxed{3\text{-}24}$

Choose $\Psi = 2\pi \Rightarrow t = \tau$ = period of the ellipse

$$\therefore \tau = 2\pi a^{3/2} (m/k)^{1/2}$$

$$\Rightarrow \tau^2 \propto a^3$$

$\Leftrightarrow$ Kepler's third law. Note the caveat

$$m = \frac{m_1 m_2}{m_1 + m_2}$$

Strictly speaking $\tau^2 \propto m a^3$

$$\therefore \tau^2 \propto \left(\frac{m_1 m_2}{m_1 + m_2}\right) a^3$$

$$m_2/m_1 << 1 \Rightarrow \tau^2 \propto m_1 a^3$$

m_1 = solar mass is same $\forall$ planets.

Kepler's 2^{ND} law $\frac{dA}{dt}$ = constant already proved earlier.

A conserved vector: $\longrightarrow$