The Lagrangian

$$L = L(\{q_i\}, \{\dot{q}_i\}, t)$$

We need the Hamiltonian

$$H = H(\{q_i\}, \{p_i\}, t) \longrightarrow 2.431$$

where $p_i \equiv \dfrac{\partial L}{\partial \dot{q}_i} \longrightarrow 2.44$

Hence we perform a LT on L.
So we define

$$H = \sum_i (\dot{q}_i p_i) - L$$

$$\therefore dH = \sum_i (\dot{q}_i dp_i + p_i d\dot{q}_i) - dL \longrightarrow 8.155$$

$$dL = \sum_i \left[\left(\frac{\partial L}{\partial q_i}\right) dq_i + \left(\frac{\partial L}{\partial \dot{q}_i}\right) d\dot{q}_i\right] + \left(\frac{\partial L}{\partial t}\right) dt \longrightarrow 8.13$$

Equations 2.44, 8.155 & 8.13 give

$$dH = \sum_i \left[\dot{q}_i dp_i + \left(\frac{\partial L}{\partial q_i}\right) dq_i(-1)\right] - \left(\frac{\partial L}{\partial t}\right) dt$$

Also by Lagrange's Equations we get

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \dot{p}_i$$

$$\therefore dH = \sum_i \left[\dot{q}_i \, dp_i - \dot{p}_i \, dq_i\right] - \left(\frac{\partial L}{\partial t}\right) dt \longrightarrow (8.16)$$

Result (2.431) $\Longrightarrow$

$$dH = \left(\frac{\partial H}{\partial t}\right) dt + \sum_i \left[\left(\frac{\partial H}{\partial q_i}\right) dq_i + \left(\frac{\partial H}{\partial p_i}\right) dp_i\right] \longrightarrow (8.17)$$

Comparing (8.17) to (8.16) we get

$$\left.\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i}, \\ \dot{p}_i &= -\left(\frac{\partial H}{\partial q_i}\right) \end{aligned}\right\} \rightarrow (8.18)$$

and $\dfrac{\partial H}{\partial t} = -\left(\dfrac{\partial L}{\partial t}\right) \longrightarrow (8.19)$

Note that (8.18) and (8.19) are $2n+1$ first order differential equations

Equations (8.18) are called Hamiltons equations.

Hamiltons equations (8.18) are $2n$ first order ones. Lagranges equations are n second order ones.

Consider

$$L(\{q_i\},\{\dot{q}_i\},t) = L_0(\{q_i\},t) + \sum_i \dot{q}_i\, a_i(\{q_i\},t)$$

$$+ \sum_i \dot{q}_i^{\,2}\, \frac{T_i}{2}(\{q_i\},t)$$

which may be written in matrix form as

$$L(\{q_i\},\{\dot{q}_i\},t) = L_0(\{q_i\},t) + \dot{\bar{q}}^T \cdot \bar{a}$$

$$+ \frac{1}{2}\dot{\bar{q}}^T \,\overline{\overline{T}}\, \dot{\bar{q}}$$

where $\bar{q} \equiv \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}_{n\times 1}$ a $n\times 1$ column matrix or vector.

$\dot{\bar{q}}^T \equiv \begin{bmatrix} \dot{q}_1 & \dot{q}_2 & \cdots & \dot{q}_n \end{bmatrix}_{1\times n}$ a $1\times n$ row matrix or transpose of a column vector.

$\overline{\overline{T}} \equiv \begin{bmatrix} T_1 & 0 & 0 & \cdots & 0 \\ 0 & T_2 & 0 & \cdots & 0 \\ & & \ddots & & \\ & & & & T_n \end{bmatrix}_{n\times n}$ a $n\times n$ matrix.

Then for such an L we get

$$H = \dot{\bar{q}}^T \cdot \bar{p} - \frac{1}{2}\dot{\bar{q}}^T \cdot \bar{\bar{T}} \cdot \dot{\bar{q}} - L_0 - \dot{\bar{q}}^T \cdot \bar{a}$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = a_i + \dot{q}_i T_i$$

$$\therefore \quad \bar{p} = \bar{a} + \bar{\bar{T}} \cdot \dot{\bar{q}}$$

$$\Rightarrow \quad \dot{\bar{q}} = \bar{\bar{T}}^{-1} \cdot (\bar{p} - \bar{a})$$

$$\dot{\bar{q}}^T = (\bar{p}^T - \bar{a}^T) \cdot (\bar{\bar{T}}^{-1})^T$$

$$= (\bar{p}^T - \bar{a}^T) \cdot \bar{\bar{T}}^{-1}$$

$$\therefore \quad H = \frac{1}{2}(\bar{p}^T - \bar{a}^T) \cdot \bar{\bar{T}}^{-1} \cdot (\bar{p} - \bar{a}) - L_0(\{q_i\}, t)$$

$$\rightarrow (8.27)$$

With result (8.27) we have now obtained $H = H(\{q_i\}, \{p_i\}, t)$

or $H = H(\bar{q}, \bar{p}, t)$.

$$\bar{\bar{T}}^{-1} = \frac{\bar{\bar{T}}_c^{\,T}}{|\bar{\bar{T}}|}$$

where $|\bar{\bar{T}}| \equiv$ determinant of $\bar{\bar{T}}$.

$\overline{\overline{T}}_c \equiv$ is the co-factor matrix whose elements $\left(\overline{\overline{T}}_c\right)_{ij} = (-1)^{j+i} \times$ the determinant of the matrix obtained by striking out the i^{TH} row and j^{TH} column.

Simplectic form of Hamiltons equations.

Consider n independent generalized coordinates $\{q_i\}$.

Let $\eta_i = q_i$, $\eta_{i+n} = p_i$, $\forall i = 1, \ldots n$.

Define a column matrix

$$\frac{\partial H}{\partial \overline{\eta}} \text{ so that } \left(\frac{\partial H}{\partial \overline{\eta}}\right)_i = \frac{\partial H}{\partial q_i}$$

and $\left(\frac{\partial H}{\partial \overline{\eta}}\right)_{i+n} = \left(\frac{\partial H}{\partial p_i}\right)$, $\forall i = 1, \ldots n$

Let $\overline{\overline{J}}$ be a $2n \times 2n$ square matrix

$$\ni \overline{\overline{J}} \equiv \begin{bmatrix} \overline{\overline{0}} & \overline{\overline{I}} \\ -\overline{\overline{I}} & \overline{\overline{0}} \end{bmatrix} \text{ where } \left(\overline{\overline{I}}\right)_{ij} = \delta_{ij}$$

and $\left(\overline{\overline{0}}\right)_{ij} = 0$, $\forall i, j$.

$$\therefore \quad \bar{\bar{J}}^T = \begin{bmatrix} \bar{\bar{0}} & -\bar{\bar{I}} \\ \bar{\bar{I}} & \bar{\bar{0}} \end{bmatrix}$$

$$\therefore \quad \bar{\bar{J}}^T \bar{\bar{J}} = \bar{\bar{J}}\bar{\bar{J}}^T = \begin{bmatrix} \bar{\bar{I}} & \bar{\bar{0}} \\ \bar{\bar{0}} & \bar{\bar{I}} \end{bmatrix}$$

$$\Rightarrow \quad \bar{\bar{J}}^T = \bar{\bar{J}}^{-1} = -\bar{\bar{J}}$$

$$\Rightarrow \quad \bar{\bar{J}}^2 = \bar{\bar{J}}\cdot\bar{\bar{J}} = -\bar{\bar{I}}$$

$$|\bar{\bar{J}}| = 1$$

Now Hamilton's equations maybe written as

$$\dot{\eta} = \bar{\bar{J}}\cdot\left(\frac{\partial H}{\partial \bar{\eta}}\right)$$

Conservation Theorems and Symmetries of H.

$$\frac{dH}{dt} = \dot{H} = \frac{\partial H}{\partial t} + \sum_i \left[\left(\frac{\partial H}{\partial q_i}\right)\dot{q}_i + \left(\frac{\partial H}{\partial p_i}\right)\dot{p}_i\right]$$

which using results (8.18) gives

$$\dot{H} = \frac{\partial H}{\partial t} = \frac{-\partial L}{\partial t}.$$

Also $H \equiv \sum_i \dot{q}_i p_i - L$

$$\Rightarrow \frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i}$$

$\Rightarrow$ If q_j is a cyclic co-ordinate of L then it is also a cyclic coordinate of H.

So all the relationships between cyclic co-ordinates and conserved quatities derived for L also hold true for H.

We now derive Hamilton's equations from a variational principle.

We need $\delta I = 0$

where
$$I = \int_{t_1}^{t_2} L(\{q_i\}, \{\dot{q}_i\}, t)\, dt$$

$$= \int_{t_1}^{t_2} \left[\sum_i p_i \dot{q}_i - H(\{q_i\}, \{p_i\}, t) \right] dt$$

Consider an infinitesimal variation parameter $\alpha \ni q_i(t, \alpha) = q_i(t, 0) + \alpha \eta_i(t)$

$\forall i$

The real trajectory i.e. the one that extremizes I is $q_i(t,0)$.

$\{\eta_i(t)\}$ are arbitrary functions.

With these definitions we get

$$\delta I = \left(\frac{\partial I}{\partial \alpha}\right) d\alpha$$

$$= (d\alpha)\frac{\partial}{\partial \alpha}\int_{t_1}^{t_2}\left[\left(\sum_i p_i \dot{q}_i\right) - H\right] dt.$$

t_1 and t_2 are fixed

$$\Rightarrow \delta I = (d\alpha)\int_{t_1}^{t_2}\sum_i \frac{\partial}{\partial \alpha}\left[\sum_i p_i \dot{q}_i - H\right] dt$$

$$= (d\alpha)\sum_i \int_{t_1}^{t_2}\left\{\left[\dot{q}_i - \frac{\partial H}{\partial p_i}\right]\left(\frac{\partial p_i}{\partial \alpha}\right) - \left(\frac{\partial H}{\partial q_i}\right)\left(\frac{\partial q_i}{\partial \alpha}\right)\right\} dt$$

$+ I_2 \, d\alpha$ where

$$I_2 \equiv \sum_i \int_{t_1}^{t_2} p_i \left(\frac{\partial \dot{q}_i}{\partial \alpha}\right) dt = \sum_i \int_{t_1}^{t_2} p_i \frac{d}{dt}\left(\frac{\partial q_i}{\partial \alpha}\right) dt$$

$$= \sum_i p_i \frac{\partial q_i}{\partial \alpha}\Bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2}\sum_i \dot{p}_i \left(\frac{\partial q_i}{\partial \alpha}\right) dt$$

$$= -\int_{t_1}^{t_2}\sum_i \dot{p}_i \left(\frac{\partial q_i}{\partial \alpha}\right) dt, \quad \because \left(\frac{\partial q_i}{\partial \alpha}\right) = \eta_i(t)$$

and $\eta_i(t) = 0$, $\forall\, t = t_1$ or t_2.

$$\therefore\ I_2\, d\alpha = -(d\alpha) \int_{t_1}^{t_2} \sum_i p_i \left(\frac{\partial q_i}{\partial \alpha}\right) dt$$

$$\therefore\ \delta I = (d\alpha) \int_{t_1}^{t_2} \sum_i \Big\{ \left[\dot{q}_i - \frac{\partial H}{\partial p_i}\right]\left(\frac{\partial p_i}{\partial \alpha}\right)$$

$$+ \left[-\dot{p}_i - \frac{\partial H}{\partial q_i}\right]\left(\frac{\partial q_i}{\partial \alpha}\right) \Big\}\, dt$$

$$= \int_{t_1}^{t_2} \sum_i \Big\{ \left[\dot{q}_i - \frac{\partial H}{\partial p_i}\right] \delta p_i + \left[-\dot{p}_i - \frac{\partial H}{\partial q_i}\right] \delta q_i \Big\}\, dt$$

where $\delta p_i \equiv \left(\frac{\partial p_i}{\partial \alpha}\right) d\alpha$ and

$\delta q_i \equiv \left(\frac{\partial q_i}{\partial \alpha}\right) d\alpha$

Since δq_i and δp_i are all independent and arbitrary variations we get

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

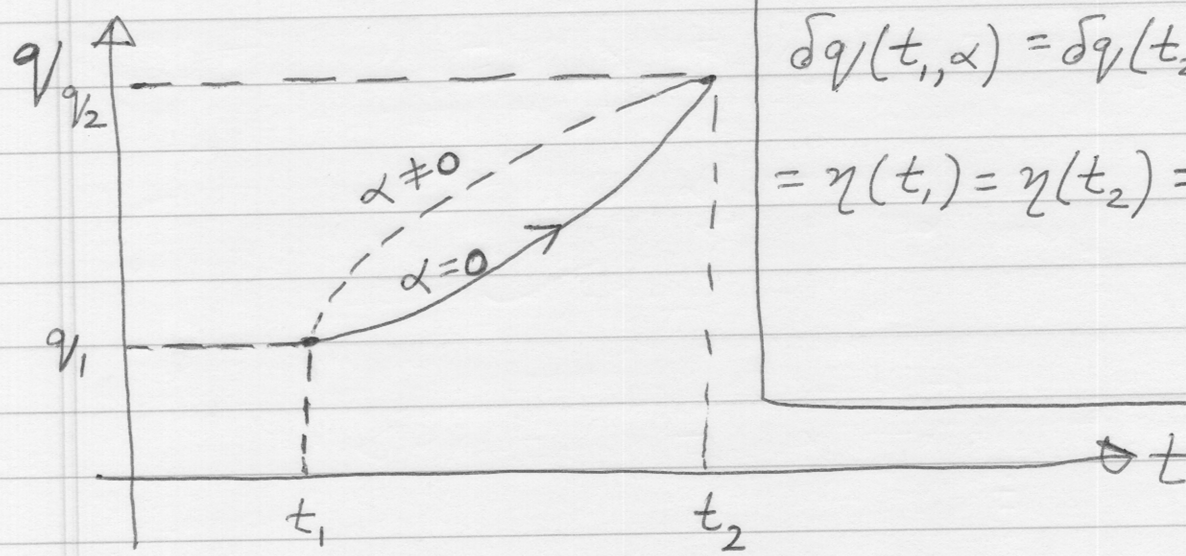
These are Hamilton's equations.

The principle of least action :→
It states that

$$\Delta \int_{t_0}^{t_2} p_i \dot{q}_i \, dt = 0$$

where Δ is a new type of variation
This is different from the δ variation
of chapter 2,

δ variation :→ We impose here that



$$\delta q(t_1, \alpha) = \delta q(t_2, \alpha)$$

$$= \eta(t_1) = \eta(t_2) = 0, \quad \forall \alpha.$$

$$q(t, \alpha) = q(t, 0) + \alpha \eta(t)$$

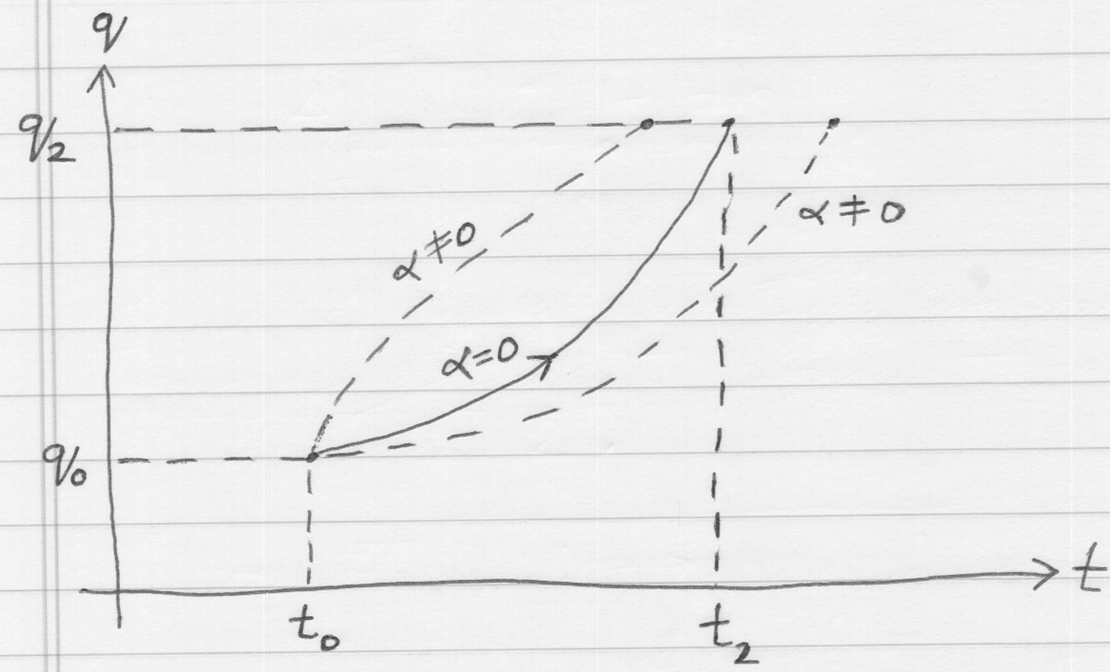
$\alpha = 0$ gives the real trajectory,

$$\delta q(t, \alpha) \equiv \left(\frac{\partial q}{\partial \alpha}\right) d\alpha = \eta(t)$$

$$\dot{q}(t, \alpha) = \alpha \dot{\eta}(t)$$

$$\delta \dot{q}(t, \alpha) \equiv \left(\frac{\partial \dot{q}}{\partial \alpha}\right) d\alpha = \dot{\eta} \, d\alpha$$

Δ variation : $\longrightarrow$



Now we have

$$q(t(\alpha), \alpha) = q(t(0), 0) + \alpha \eta(t(\alpha))$$

$$\Delta q(t(\alpha), \alpha) \equiv \left[\frac{dq(t(\alpha), \alpha)}{d\alpha}\right] d\alpha$$

$$= \left[\frac{\partial q}{\partial \alpha} + \frac{\partial q}{\partial t}\left(\frac{dt}{d\alpha}\right)\right] d\alpha$$

$$= \left[\eta(t(\alpha)) + \alpha \dot{\eta}(t(\alpha)) \times \frac{dt}{d\alpha}\right] d\alpha$$

$\Rightarrow$ $\Delta q \equiv \delta q + \dot{q}\,\Delta t$, where $\Delta t \equiv \left(\frac{dt}{d\alpha}\right) d\alpha$.

$\longrightarrow$ (8.76)

We now impose

$$\Delta q(t_0(\alpha), \alpha) = \delta q(t(\alpha), \alpha) = \Delta t(\alpha)\Big|_{t_0} = 0$$

and

$$\Delta q(t_2(\alpha), \alpha) = 0 \ , \ \forall \alpha$$

Note that $\Delta q(t_2(\alpha), \alpha) = \delta q(t_2(\alpha), \alpha) + \Delta t\Big|_{t_2}$

$\Rightarrow \delta q(t_2(\alpha), \alpha) \neq 0$ in general.

We impose also that H is conserved for all possible variations

i.e $\quad H(\{p_i(t(\alpha), \alpha)\}, \{q_i(t(\alpha), \alpha)\}, t)$

$$= \text{constant} \ , \ \forall \alpha .$$

Now for any function $f = f(\{q_i\}, t)$

we get

$$\Delta f = \sum_i \left(\frac{\partial f}{\partial q_i}\right) \Delta q_i + \left(\frac{\partial f}{\partial t}\right) \Delta t$$

$$= \sum_i \left(\frac{\partial f}{\partial q_i}\right) \delta q_i + \sum_i \left(\frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial t}\right) \Delta t$$

$$= \delta f + \dot{f} \Delta t \longrightarrow (8.765)$$

Thus we need 3 conditions to prove the principle of least action.

8-15

(i) $\Delta q_i(t_2(\alpha), \alpha) = 0 \quad , \forall \alpha$

(ii) $H(\{p_i(t(\alpha),\alpha)\}, \{q_i(t(\alpha),\alpha)\}, t) = \text{constant} \quad \forall \alpha$

(iii) Lagrange's equations are true or that Hamiltons or D'Alemberts principle hold true.

Proof : $\rightarrow$ $$\Delta S \equiv \Delta \int_{t_0}^{t_2} p_i \dot{q}_i \, dt$$

$$\therefore \Delta S = \Delta \int_{t_0}^{t_2} (L + H) dt = \Delta \int_{t_0}^{t_2} L dt + H(\Delta t) \longrightarrow 8.767$$

because we used (ii).

Now using 8.765 we get

$$\Delta \int_{t_0}^{t_2} L dt = \delta \int_{t_0}^{t_2} L dt + \left[\frac{d}{dt_2} \int_{t_0}^{t_2} L dt \right] \Delta t$$

$$= \delta \int_{t_0}^{t_2} L dt + L(\Delta t) \longrightarrow 8.766$$

$\therefore$ 8.767 & 8.766 together give

$$\Delta S = (\Delta t)[L + H] + \delta \int_{t_0}^{t_2} L\,dt \longrightarrow 8.769$$

Note that the last term though similar to the one in Hamiltons principle is not zero. This is because while evaluating it we need to remember that $\Delta q_i(t_2(\alpha), \alpha) = 0, \ \forall \alpha$

but $\delta q_i(t_2(\alpha), \alpha) \neq 0$.

Note H is conserved $\Rightarrow \dot{H} = \frac{\partial L}{\partial t} = 0$

$$\Rightarrow \quad \delta L = \sum_i \left[\left(\frac{\partial L}{\partial q_i}\right) \delta q_i + \left(\frac{\partial L}{\partial \dot{q}_i}\right) \delta \dot{q}_i \right]$$

$$= \sum_i \left\{ \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right] \delta q_i + \left(\frac{\partial L}{\partial \dot{q}_i}\right) \frac{d}{dt}(\delta q_i) \right\}$$

$$= \sum_i \frac{d}{dt}\left[\left(\frac{\partial L}{\partial \dot{q}_i}\right) \delta q_i\right]$$

$$\therefore \delta \int_{t_0}^{t_2} L\,dt = \int_{t_0}^{t_2} (\delta L)\,dt = \sum_i \int_{t_0}^{t_2} d\left(\left(\frac{\partial L}{\partial \dot{q}_i}\right) \delta q_i\right)$$

$$= \sum_i \left(\frac{\partial L}{\partial \dot{q}_i}\right) \delta q_i \Big|_{t_0}^{t_2} = \sum_i \left(\frac{\partial L}{\partial \dot{q}_i}\right)(\Delta q_i - \dot{q}_i \Delta t) \Big|_{t_0}^{t_2}$$

$$= -\sum_i \left(\frac{\partial L}{\partial \dot{q}_i}\right) \dot{q}_i \, \Delta t \Big|_{t_0}^{t_2}$$

$$= -\sum_i p_i(t_2)\, \dot{q}_i(t_2)\, \Delta t$$

$$\therefore \quad \delta \int_{t_0}^{t_2} L\, dt = -\sum_i p_i(t_2)\, \dot{q}_i(t_2)\, \Delta t \quad \longrightarrow (8.768)$$

$\therefore$ (8.768) & (8.769) give

$$\Delta S = \Delta t \left[L + H - \sum_i p_i \dot{q}_i \right]$$

$$= 0 \quad \text{by definition of } H.$$

Proof ends.

Now consider a special case $V = V(\{q_i\})$ and

$$T = \sum_{i,j} \tfrac{1}{2} M_{ji}\, \dot{q}_i \dot{q}_j$$

$$\Rightarrow \sum_i \dot{q}_i p_i = \sum_i \dot{q}_i \frac{\partial H}{\partial \dot{q}_i} = 2T$$

$$\therefore \quad \Delta S = 0 \Rightarrow \Delta \int_{t_0}^{t_2} 2T\, dt = 0$$

Also consider that there are no external forces on the system.

Then $T =$ constant 8-18

$$\Rightarrow 2T \Delta \int_{t_0}^{t_2} dt = 0$$

$$\Rightarrow \Delta(t_2 - t_0) = 0$$

$\Rightarrow$ the time to go from one point in phase space to another is a minimum.