The ~~tee~~ term $\sum_{\alpha=1}^{m} \lambda_\alpha \frac{\partial f_\alpha}{\partial \dot{q}_j}$ is similar

to the term Q_j' of (1.665).

Hence we identify it with the generalized forces causing the constraint.

The advantages of the variational principle is that many physical laws can be stated as variational principles.

Symmetry $\Longleftrightarrow$ Conservation Law.

Consider $V \equiv V(\{q_i\})$

$$\Rightarrow \frac{\partial V}{\partial \dot{q}_i} = 0 \quad , \text{ Let } q_i \equiv x_i , \quad i = 1,2,3$$
$$\equiv (x, y, z)$$

$$\Rightarrow \frac{\partial L}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}} \quad \text{(say)}$$

$= m\dot{x} =$ linear momentum along x.

In general canonical or conjugate momentum $\equiv p_j \equiv \frac{\partial L}{\partial \dot{q}_j}$

If L has no dependence on q_j then q_j is called a cyclic or ignorable coordinate.

If q_j is cyclic then

$$\left[\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j} - \frac{\partial}{\partial q_j}\right] L = \frac{dp_j}{dt} = 0$$

$$\Rightarrow p_j \equiv \frac{\partial L}{\partial \dot{q}_j} = \text{constant.}$$

Another way of saying this:

IF

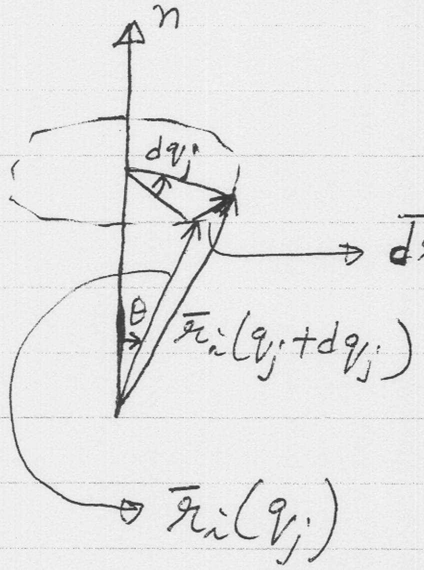
$$\left\{ L(q_1, q_2, \ldots q_i, q_{i+1}, \ldots q_n, \dot{q}_1, \dot{q}_2, \ldots \dot{q}_i, \ldots \dot{q}_n, t) = L(q_1, q_2, \ldots q_i + \Delta, q_{i+1}, \ldots q_n, \dot{q}_1, \dot{q}_2, \ldots \dot{q}_i \ldots \dot{q}_n, t) \right\}$$

THEN

$$\left\{ p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \text{ is conserved} \right\}$$

$\therefore$ Generalized translational symmetry in q_i of L $\Longleftrightarrow$ Conservation of generalized momentum p_i.

Consider Fig 2.8



$$\therefore \ |\overline{dr}_i| = |\bar{r}_i| \sin\theta \, dq_j$$

$$\left|\frac{\partial \bar{r}_i}{\partial q_j}\right| = r_i \sin\theta$$

$$\Rightarrow \quad \frac{\partial \bar{r}_i}{\partial q_j} = \hat{n} \times \bar{r}_i \quad , \ \hat{n} = \text{unit vector along rotation axis.}$$

$$\therefore \ Q_j = \sum_i \bar{F}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j} = \sum_i \bar{F}_i \cdot (\hat{n} \times \bar{r}_i)$$

$$= \sum_i (\bar{r}_i \times \bar{F}_i) \cdot \hat{n} = \hat{n} \cdot \sum_i \bar{N}_i = \hat{n} \cdot \bar{N}$$

where $\bar{N}_i \equiv \bar{r}_i \times \bar{F}_i$

$$\bar{N} \equiv \sum_i \bar{N}_i$$

Now $$p_j = \frac{\partial T}{\partial \dot{q}_j} = \sum_i m_i \bar{v}_i \cdot \left(\frac{\partial \bar{v}_i}{\partial \dot{q}_j}\right)$$

$$= \sum_i m_i \bar{v}_i \cdot \left(\frac{\partial \bar{r}_i}{\partial q_j}\right) \qquad \text{(using (1.51))}.$$

$$\therefore \; p_j = \sum_i m_i \bar{v}_i \cdot (\hat{n} \times \bar{r}_i)$$

$$= \sum_i m_i (\bar{r}_i \times \bar{v}_i) \cdot \hat{n} = \hat{n} \cdot \sum_i \bar{L}_i = \hat{n} \cdot \bar{L}$$

where $$\bar{L}_i \equiv m_i \bar{r}_i \times \bar{v}_i$$

$$\bar{L} \equiv \sum_i \bar{L}_i$$

If q_j is cyclic when it is purely a rotation coordinate then

$$\dot{q}_j = 0 \Rightarrow \frac{dp_j}{dt} = 0$$

$$\Rightarrow \hat{n} \cdot \bar{L} = \text{constant}.$$

$\therefore$ If L is invariant under certain rotations then the angular momentum is conserved under those rotations.

Consider now the case of equation (1.575)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j'$$

$Q_j' \equiv$ non-conservative part of the generalized force.

Define $h(\{q_i\},\{\dot{q}_i\},t) \equiv \left(\sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) - L$

and let

$$L \equiv L(\{q_i\},\{\dot{q}_i\},t)$$

$$\therefore \quad \dot{h} \equiv \frac{dh}{dt} = \left[\sum_j \ddot{q}_j \frac{\partial L}{\partial \dot{q}_j} + \dot{q}_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\right] - \frac{dL}{dt}$$

From (1.575) we get

$$\dot{h} = -\frac{dL}{dt} + \sum_j\left(\ddot{q}_j \frac{\partial L}{\partial \dot{q}_j} + \dot{q}_j \frac{\partial L}{\partial q_j}\right)$$

$$+ \sum_j \dot{q}_j Q_j'$$

$$= -\frac{dL}{dt} + \left\{\frac{dL}{dt} - \frac{\partial L}{\partial t}\right\} + \sum_j \dot{q}_j Q_j'$$

$$\therefore \dot{h} = -\frac{\partial L}{\partial t} + \sum_j \dot{q}_j Q_j'$$

h is called the energy function.
h is not <u>necessarily</u> the total energy of the system.

Definition $L \equiv T - V$ is co-ordinate independent

Definition of $h \equiv \left(\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right) - L$ depends on $\{q_i\}$.

$\dot{h} = 0$ or h = constant

if and only if (iff)

(i) $\frac{\partial L}{\partial t} = 0$ & $\dot{q}_j Q_j' = 0$, $\forall j$

i.e. (i) $L = L(\{q_i\}, \{\dot{q}_i\})$ & $\dot{q}_j Q_j' = 0$, $\forall j$.

When is $h = T + V$?

Now consider the case $\exists$

(ii) $V = V(\{q_i\}) \Rightarrow \frac{\partial V}{\partial \dot{q}_j} = 0$

and (iii) $\bar{r}_i = \bar{r}_i(\{q_i\})$ i.e. no explicit time dependence in going from $\bar{r}_i \to q_i$

Under conditions (ii) – (iii) we get

$$h = \sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L$$

$$= \sum_k \dot{q}_k \frac{\partial T}{\partial \dot{q}_k} - T + V$$

$$= \sum_k \dot{q}_k \frac{\partial}{\partial \dot{q}_k} \sum_i \frac{m_i}{2} \dot{\bar{r}}_i^{\,2} - T + V$$

$$= V - T + \sum_k \dot{q}_k \sum_i m_i \dot{\bar{r}}_i \cdot \left(\frac{\partial \dot{\bar{r}}_i}{\partial \dot{q}_k}\right) \longrightarrow 2.564$$

Now $\bar{r}_i = \bar{r}_i(\{q_i\})$

$$\Rightarrow \dot{\bar{r}}_i = \sum_j \dot{q}_j \frac{\partial \bar{r}_i}{\partial q_j}$$

$$\Rightarrow \frac{\partial \dot{\bar{r}}_i}{\partial \dot{q}_k} = \sum_j \left(\frac{\partial \bar{r}_i}{\partial q_j}\right) \delta_{jk} = \frac{\partial \bar{r}_i}{\partial q_k}$$

$$\Rightarrow \dot{\bar{r}}_i = \sum_j \dot{q}_j \frac{\partial \bar{r}_i}{\partial q_j} = \sum_j \dot{q}_j \frac{\partial \dot{\bar{r}}_i}{\partial \dot{q}_j}$$

$$\Rightarrow \dot{\bar{r}}_i = \sum_k \dot{q}_k \frac{\partial \dot{\bar{r}}_i}{\partial \dot{q}_k} \longrightarrow 2.565$$

Combining 2.564 and 2.565 we get

$$h = V - T + \sum_i m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i$$

$$= V - T + 2 \sum_i \frac{m_i}{2} \vec{v}_i^{\,2}$$

$$= V - T + 2T$$

$\therefore$ $h = T + V$ when conditions (ii) – (iii) are satisfied.

Now consider $\dfrac{\partial L}{\partial t} = 0$

and $Q_j' = -\dfrac{\partial \mathcal{F}}{\partial \dot{q}_j}$

Then $\dfrac{dh}{dt} = -\sum_j \dot{q}_j \dfrac{\partial \mathcal{F}}{\partial \dot{q}_j} = -2\mathcal{F}$

by Euler's Theorem, which states that

$$\sum_i x_i \frac{\partial f}{\partial x_i}(\{x_i\}) = nf$$

for a homogneous function f of degree n.

Homogeneous means

$$f(\{\lambda x_i\}) = \lambda^n f(\{x_i\}).$$

Chapter 8: Hamilton's Equations

Legendre Transformation

Consider $f = f(x, y)$

$$\Rightarrow \quad df = u(x,y)\,dx + v(x,y)\,dy$$

where $u(x,y) \equiv \dfrac{\partial f(x,y)}{\partial x}$

$$v(x,y) \equiv \frac{\partial f(x,y)}{\partial y}$$

We want a function $g(u,y)$ which has the same information as $f(x,y)$
If we choose

$$g(u,y) = f - ux$$

we get
$$dg = df - u\,dx - x\,du$$
$$= v\,dy - x\,du$$

$$\Rightarrow \quad v = \frac{\partial g(u,y)}{\partial y} \quad \text{and} \quad x = -\frac{\partial g(u,y)}{\partial u}$$

g is called the Legendre Transform (LT) of f.

In thermodynamics we have

$$U = U(S, V)$$

$$\Rightarrow \quad dU = \left(\frac{\partial U}{\partial S}\right) dS + \left(\frac{\partial U}{\partial V}\right) dV$$
$$= \quad T\,dS - P\,dV$$

where we have the definitions

$$T \equiv \left(\frac{\partial U}{\partial S}\right)_V \quad , \quad P = \left(-\frac{\partial U}{\partial V}\right)_S$$

The enthalpy $H = U + PV$ is the LT of

$$U = U(S, V)$$

$$\therefore \quad H = H(S, P)$$

$$dH = dU + P\,dV + V\,dP$$
$$= T\,dS + V\,dP$$

with $T \equiv \left(\frac{\partial H}{\partial S}\right)_P \quad , \quad V \equiv \left(\frac{\partial H}{\partial P}\right)_S$

Similarly Helmholtz free energy

$$F = U - TS \quad , \quad \Rightarrow F = F(T, V)$$

& Gibb's free energy

$$G = H - TS \quad , \quad G = G(T, P)$$
$$= U + PV - TS$$