# Summary -- Translation - Rotation

<table>
<thead>
<tr>
<th>Translational motion</th>
<th>Quantity</th>
<th>Rotational motion</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>Position</td>
<td>θ</td>
</tr>
<tr>
<td>Δx</td>
<td>Displacement</td>
<td>Δθ</td>
</tr>
<tr>
<td>v = dx/dt</td>
<td>Velocity</td>
<td>ω = dθ/dt</td>
</tr>
<tr>
<td>a = dv/dt</td>
<td>Acceleration</td>
<td>α = dω/dt</td>
</tr>
<tr>
<td>m</td>
<td>Mass Inertia</td>
<td>I</td>
</tr>
<tr>
<td>F = ma</td>
<td>Newton’s second law</td>
<td>τ = r × F</td>
</tr>
<tr>
<td>W = ( \int_{x_i}^{x_f} \vec{F} \cdot d\vec{x} )</td>
<td>Work</td>
<td>W = ( \int_{\theta_i}^{\theta_f} \tau \ d\theta )</td>
</tr>
<tr>
<td>K = ( \frac{1}{2} ) mv(^2)</td>
<td>Kinetic energy</td>
<td>K = ( \frac{1}{2} ) Iω(^2)</td>
</tr>
<tr>
<td>P = ( \vec{F} \cdot \vec{v} )</td>
<td>Power (constant ( \vec{F} ) or ( \tau ))</td>
<td>P = τω</td>
</tr>
</tbody>
</table>
The Kinetic Energy of Rolling

View the rolling as pure rotation around P, the kinetic energy

\[ K = \frac{1}{2} I_P \omega^2 \]

parallel axis theorem: \( I_P = I_{\text{com}} + MR^2 \)

so \[ K = \frac{1}{2} I_{\text{com}} \omega^2 + \frac{1}{2} MR^2 \omega^2 \]

since \( v_{\text{com}} = \omega R \)

\[ K = \frac{1}{2} I_{\text{com}} \omega^2 + \frac{1}{2} M(v_{\text{com}})^2 \]

\( \frac{1}{2} I_{\text{com}} \omega^2 \): due to the object’s rotation about its center of mass

\( \frac{1}{2} M(v_{\text{com}})^2 \): due to the translational motion of its center of mass
Chapter 11: Rolling, Torque, and Angular Momentum

For an object rolling smoothly, the motion of the center of mass is pure translational.

\[ s = \theta R \]

\[ v_{\text{com}} = \frac{ds}{dt} = \frac{d(\theta R)}{dt} = \omega R \]

\[ v_{\text{com}} = \omega R \]
• Rolling viewed as a combination of pure rotation and pure translation

\[ v_{\text{top}} = (\omega)(2R) = 2 v_{\text{com}} \]

• Rolling viewed as pure rotation

• Different views, same conclusion
Sample Problem: A uniform solid cylindrical disk, of mass \( M = 1.4 \) kg and radius \( R = 8.5 \) cm, rolls smoothly across a horizontal table at a speed of 15 cm/s. What is its kinetic energy \( K \)?

\[
v_{\text{c.m.}} = 0.15 \text{m/s}
\]

\[
I_{\text{disk}} = \frac{1}{2}MR^2 = (0.5)(1.4\text{kg})(0.085\text{m})^2 = 5.058 \times 10^{-3}\text{kg m}^2
\]

\[
\omega = \frac{v}{R} = \frac{0.15\text{m/s}}{0.085\text{m}} = 1.765 \text{ rad/s}
\]

\[
K = K_{\text{trans}} + K_{\text{rot}} = \frac{1}{2}Mv_{\text{c.m.}}^2 + \frac{1}{2}I\omega^2
\]

\[
K = K_{\text{trans}} + K_{\text{rot}} = \frac{1}{2}(1.4)(0.15)^2 + \frac{1}{2}(5.058 \times 10^{-3})(1.765)^2
\]

\[
= 15.75 \times 10^{-3} + 7.878 \times 10^{-3} = 23.63 \times 10^{-3} \text{J}
\]
Angular momentum

Angular momentum with respect to point O for a particle of mass $m$ and linear momentum $p$ is defined as:

$$\vec{\ell} = \vec{r} \times \vec{p} = m (\vec{r} \times \vec{v})$$

Compare to the linear case $\vec{p} = m\vec{v}$

direction: right-hand rule

magnitude:

$$\ell = r p \sin \phi = r mv \sin \phi$$
The angular momentum of a rigid body rotating about a fixed axis

Consider a simple case, a mass $m$ rotating about a fixed axis $z$:

$$\ell = r \, mv \, \sin 90^\circ = r \, m \, r \, \omega = mr^2 \omega = I \omega$$

In general, the angular momentum of a rigid body rotating about a fixed axis is

$$L = I \, \omega$$

$L$ : angular momentum (group or body) along the rotation axis

$\ell$ : angular momentum (particle) along the rotation axis

$I$ : moment of inertia about the same axis
Question

Particles 1, 2, 3, 4, and 5 have the same mass and speed as shown in the figure. Particles 1 & 2 move around $O$ in opposite directions. Particles 3, 4, and 5 move towards or away from $O$ as shown.

Which of the particles has the smallest magnitude angular momentum?

1) 1  2) 2  3) 3  4) 4  5) 5  6) all have the same $l$
Question

Particle with smallest angular momentum (magnitude)?

1. 1
2. 2
3. 3
4. 4
5. 5
6. all have same ang. mom.
Question

Particles 1, 2, 3, 4, and 5 have the same mass and speed as shown in the figure. Particles 1 & 2 move around $O$ in opposite directions. Particles 3, 4, and 5 move towards or away from $O$ as shown.

\[ l = r p \sin \phi = r m v \sin \phi \quad \phi = 0^\circ \text{ for } 5. \implies l = 0 \]

Which of the particles has the smallest magnitude angular momentum?

1) 1  2) 2  3) 3  4) 4  5) 5  6) all have the same $l$
Newton’s Second Law in Angular Form

$$\vec{\tau} = I\vec{\alpha} = \vec{r} \times \vec{F} = \vec{r} \times \frac{d(m\vec{v})}{dt} = \frac{d(\vec{r} \times m\vec{v})}{dt} = \frac{d\vec{\ell}}{dt}$$

Let $\vec{\tau}_{net}$ be the vector sum of all the torques acting on the object.

$$\vec{\tau}_{net} = \frac{d\vec{L}_{\text{total}}}{dt}$$

$$\vec{L}_{\text{total}} = \sum_{i=1}^{n} \vec{l}_{i}$$

**Net external torque** equals to the **time rate change** of the system’s **total angular momentum**
Torque and Angular Momentum

\[ \vec{\tau}_{\text{net}} = \frac{d\vec{l}}{dt} = \frac{d(\vec{r} \times m\vec{v})}{dt} \]

\[ = \vec{r} \times \left( m \frac{d\vec{v}}{dt} \right) = \vec{r} \times (m\vec{a}) = \vec{r} \times \vec{F} \]

**Torque** is the time rate of change of **angular momentum**.
Precession

Torque is the time rate of change of angular momentum.

Falling due to torque about the pivot point.

\[ \tau = rF\sin\theta = rmg \sin\theta \]

Falling causes angular momentum about the pivot point (along y-axis).
Precession

Torque is the time rate of change of angular momentum.

Falling due to torque about the pivot point.

$$\tau = rF \sin \theta = rmg \sin \theta$$

Falling causes angular momentum about the pivot point (along y-axis).

Now set the gyroscope in motion

$$L = I \omega$$  (along x-axis)

$L$ is fixed by the spinning, so the torque can only change the direction of $L$
Precession Rate

Torque is the time rate of change of angular momentum.

Falling due to torque about the pivot point.

Falling causes angular momentum about the pivot point (along y-axis).

\[
\frac{d\phi}{dt} = \frac{dL}{L} = \frac{Mg r}{I\omega} \quad (\text{Precession along z-axis})
\]
Precession Rate

Torque is the time rate of change of angular momentum.

Nuclei have intrinsic angular momentum.

This effect is at the core of MRI, which is tuned to pick up the intrinsic angular momentum of the proton in hydrogen.

(Precession along z-axis)
Conservation of Angular Momentum

If the net external torque acting on a system is zero, the angular momentum of the system is conserved.

\[ \tau_{\text{net}} = 0 \quad \text{then} \quad \vec{L} = \text{constant} \]

\[ \vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt} \]

\[ \vec{L}_i = \vec{L}_f \]

For a rigid body rotating around a fixed axis, ( \( L = I \omega \) ) the conservation of angular momentum can be written as

\[ I_i \omega_i = I_f \omega_f \]
Some examples involving conservation of angular momentum

The spinning volunteer

$L_f = L_i \Rightarrow I_f \omega_f = I_i \omega_i$
Angular momentum is conserved

$L_i$ is in the spinning wheel

Now exert a torque to flip its rotation.
$L_{f,\text{wheel}} = -L_i$.

Conservation of Angular momentum means that the person must now acquire an angular momentum.
$L_{f,\text{person}} = +2L_i$
so that $L_f = L_{f,\text{person}} + L_{f,\text{wheel}} = +2L_i + -L_i = L_i$. 
More examples

The springboard diver

Spacecraft orientation
Problem 11-66

Ring of $R_1 (=R_2/2)$ and $R_2 (=0.8m)$, Mass $m_2 = 8.00kg$. 
$\omega_i = 8.00$ rad/s. Cat $m_1 = 2kg$. Find kinetic energy change when cat walks from outer radius to inner radius.
Problem 11-66

Ring of $R_1 (=R_2/2)$ and $R_2 (=0.8m)$, Mass $m_2 = 8.00$kg. 
$\omega_i = 8.00$ rad/s. Cat $m_1 = 2$kg. Find kinetic energy change when cat walks from outer radius to inner radius.

**Initial Momentum**

\[
L_i = L_{i,\text{cat}} + L_{i,\text{ring}} = m_1 R_2 v_i + I \omega_i \\
= m_1 R_2 \omega_i + \frac{1}{2} m_2 (R_1^2 + R_2^2) \omega_i \\
= m_1 R_2 \omega_i \left(1 + \frac{1}{2} \frac{m_2}{m_1} \left(\frac{R_1^2}{R_2^2} + 1\right)\right)
\]
Problem 11-66

Ring of $R_1 (= R_2/2)$ and $R_2 (=0.8\text{m})$, Mass $m_2 = 8.00\text{kg}$.  
$\omega_i = 8.00\text{ rad/s}$. Cat $m_1 = 2\text{kg}$. Find kinetic energy change when cat walks from outer radius to inner radius.

**Final Momentum**

\[
L_f = L_{f,\text{cat}} + L_{f,\text{ring}} = m_1 R_1 v_f + I \omega_f
\]

\[
= m_1 R_1^2 \omega_f + \frac{1}{2} m_2 (R_1^2 + R_2^2) \omega_f
\]

\[
= m_1 R_1^2 \omega_f \left( 1 + \frac{1}{2} \frac{m_2}{m_1} \left( \frac{R_2^2}{R_1^2} + 1 \right) \right)
\]
Problem 11-66

Then from $L_f = L_i$ we obtain

$$
\frac{\omega_f}{\omega_0} = \frac{R_2^2}{R_1^2} \frac{1 + \frac{1}{2} \frac{m_2}{m_1} \left( \frac{R_2^2}{R_1^2} + 1 \right)}{1 + \frac{1}{2} \frac{m_2}{m_1} \left( 1 + \frac{R_2^2}{R_1^2} \right)} = (2.0)^2 \frac{1 + 2(0.25 + 1)}{1 + 2(1 + 4)} = 1.273
$$

Thus, $\omega_f = 1.273 \omega_0$. Using $\omega_0 = 8.00 \text{ rad/s}$, we have $\omega_f = 10.2 \text{ rad/s}$. By substituting $I = L/\omega$ into $K = \frac{1}{2} I \omega^2$, we obtain $K = \frac{1}{2} L \omega$. Since $L_i = L_f$, the kinetic energy ratio becomes

$$
\frac{K_f}{K_i} = \frac{1}{2} \frac{L_f \omega_f}{L_i \omega_i} = \frac{\omega_f}{\omega_0} = 1.273.
$$

which implies $\Delta K = K_f - K_i = 0.273 K_i$. The cat does positive work while walking toward the center of the ring, increasing the total kinetic energy of the system.
Problem 11-66

Ring of \( R_1 (=R_2/2) \) and \( R_2 (=0.8\text{m}) \),
Mass \( m_2 = 8.00\text{kg} \).
\( \omega_i = 8.00 \text{ rad/s} \).  Cat \( m_1 = 2\text{kg} \).  Find
kinetic energy change when cat walks
from outer radius to inner radius.

Initial Kinetic energy \( K_i \) is:

\[
K_i = \frac{1}{2} \left[ m_1 R_2^2 + \frac{1}{2} m_2 (R_1^2 + R_2^2) \right] \omega_0^2 = \frac{1}{2} m_1 R_2^2 \omega_0^2 \left[ 1 + \frac{1}{2} \frac{m_2}{m_1} \left( \frac{R_1^2}{R_2^2} + 1 \right) \right]
\]

\[
= \frac{1}{2} (2.00 \text{ kg})(0.800 \text{ m})^2(8.00 \text{ rad/s})^2[1+(1/2)(4)(0.5^2+1)]
\]

\[
= 143.36 \text{ J},
\]

the increase in kinetic energy is \( \Delta K = (0.273)(143.36 \text{ J}) = 39.1 \text{ J} \).