The Thomas precession gives $g_e - 1$, not $g_e/2$

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Modern experimental techniques such as laser-rf excitation, quantum beat spectroscopy, level crossing measurements, etc. provide direct determinations of atomic fine structure splittings, often to uncertainties of less than 10^{-10} Rydberg units. For hydrogenlike atoms these splittings can be predicted by a neat and concise semiclassical development of the spin-orbit interaction energy, presented in many textbooks on atomic physics (Refs. 1–5), that is

based on the Biot–Savart law and the vector model of angular momentum. Although the data available are of sufficient accuracy that self-energy corrections to the anomalous magnetic moment of the electron are necessary to obtain agreement, these contributions are neglected in most textbook developments. The purpose of this note is to indicate how self-energy corrections can be retained in the standard exposition of the spin-orbit energy with no increase in complexity, producing an expression that agrees with the experiment to within four parts in 10^5.

The standard textbook development[1–5] of the spin-orbit interaction energy considers the magnetic field seen by an electron in a hydrogenlike atom due to the apparent motion of the nucleus, which is given by (in standard symbols[1–8])

$$\mathbf{B} = \frac{k}{c^2}\frac{Ze(\mathbf{r}\times\mathbf{v})}{r^3} = \frac{Zke}{mc^2}\frac{\mathbf{L}}{r^3}, \tag{1}$$

and the anomalous magnetic moment of the electron, given by

$$\boldsymbol{\mu}_s = -g_e\frac{e}{2m}\mathbf{S}. \tag{2}$$

In the rest frame of the electron the interaction energy for these two charge circulations is

$$\Delta E = -\boldsymbol{\mu}_s\cdot\mathbf{B} = g_e\frac{Zke^2}{2(mc)^2}\frac{\mathbf{L}\cdot\mathbf{S}}{r^3}. \tag{3}$$

To transform to the rest frame of the nucleus it is necessary to take into account the Thomas precession. This is discussed in Refs. 1, 6, and 7, and requires that the Larmor frequency due to electron spin be corrected by the addition of the transformational Thomas frequency. The Thomas precession is in a direction opposite to the electron Larmor precession, and involves all of the same quantities except for the electron g factor. The frequency after transformation to the rest system of the nucleus is

$$\boldsymbol{\omega} = g_e\frac{e\mathbf{B}}{2m} - \frac{e\mathbf{B}}{2m} = (g_e-1)\frac{e\mathbf{B}}{2m}. \tag{4}$$

Since the energy is given by

$$\Delta E = \boldsymbol{\omega}\cdot\mathbf{S} = (g_e-1)\frac{Zke^2}{2(mc)^2}\frac{\mathbf{L}\cdot\mathbf{S}}{r^3}, \tag{5}$$

the $g_e - 1$ factor also occurs in the energy. At this point most textbooks approximate the anomalous magnetic moment by the Dirac moment $g_e(D) = 2$, neglecting the electron self-energy corrections that are given by

$$g_e = 2 + (\alpha/\pi) - 0.657(\alpha/\pi)^2 + \cdots. \tag{6}$$

(Models for the origin of these corrections have been presented by Grotch and Kazes.[8]) Some textbook developments substitute $g_e(D) - 1 = 1$, removing the explicit dependence upon g_e, whereas others substitute $g_e(D) - 1 = g_e(D)/2$. Although the use of the multiplicative factor $g_e/2$ to make the additive correction for the Thomas precession is technically correct in the limit $g_e \to g_e(D) = 2$, its use can be misleading to students, since

$$g_e/2 = 1 + \alpha/2\pi + \cdots, \tag{7}$$

whereas

$$g_e - 1 = 1 + \alpha/\pi + \cdots. \tag{8}$$

Possible confusion in comparing textbook expressions describing atomic physics and quantum electrodynamics could be avoided simply by retaining the general correction factor $g_e - 1$ in the expression for the fine structure. The self-energy corrections are important, amounting to 0.23%, substantially larger than, e.g., reduced mass corrections that are only 0.05%.

If self-energy corrections are included through the $g_e - 1$ factor, and the expectation values

$$\langle\mathbf{L}\cdot\mathbf{S}\rangle = [j(j+1) - l(l+1) - s(s+1)]\hbar^2/2 \tag{9}$$

and

$$\langle r^{-3}\rangle = (Z/a_0n)^3[l(l+\tfrac{1}{2})(l+1)]^{-1} \tag{10}$$

are substituted into Eq. (5) and reduced mass corrections are made, the expression for the fine structure separation between two levels with the same n, l, s, and $j = l + s(+)$ and $j = l - s(-)$ obtained from Eqs. (5), (8)–(10) is

$$\langle\Delta E(+) - \Delta E(-)\rangle = \frac{(1+\alpha/\pi)}{(1+m_e/M_p)}\frac{R\alpha^2Z^4}{n^3l(l+1)}(2s). \tag{11}$$

For the fine structure of the $2p$ term in hydrogen, Eq. (11) yields 10 968.74 mHz, which is within four parts in 10^5 of the experimental value of 10 969.127(87) mHz.[10]

In addition to its application to hydrogenlike values, Eq. (11) is also useful in describing the fine structure of x ray and optical spectra in complex many-electron atoms, through the use of a noninteger screened charge $Z \to Z - S$. Eq. (11) is the lowest-order term in an αZ expansion of the Dirac energy, and an explicit general expression for generating higher order corrections is given in Ref. 11.

Thus, within the spirit of the heuristic derivation, this exposition of the spin-orbit interaction accounts for the self-energy corrections through the factor $g_e - 1$, and provides a useful pedagogic example which illustrates the very high precision that is possible in atomic spectroscopy.

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