INDEX NOTATION

## A. Three Types of Physical Quantities: Scalars, Vectors, and Tensors

Many problems in engineering and applied mathematics can be conveniently solved by a notational method known as vector analysis. Vector operators are defined, and a useful set of vector identities is obtained. However, in problems of theoretical physics, the methods of vector analysis become very restrictive, and a more powerful mathematics is required. In order to develop this mathematics we must be much more explicit in our definition of the properties of a vector.

A physicist bases his definition of a scalar, a vector, and a tensor upon the transformation properties of the quantity; that is, the behavior of the quantity as it undergoes a change of basis or representation (coordinates), where both the original and final bases span the space containing the quantity. Let us consider a set of orthogonal basis vectors $(x_1, x_2, x_3)$ in configuration space. Let us then transform to a new set of basis vectors $(x_1', x_2', x_3')$. Let the two sets be connected by relationships of the form

$$x_i' = \sum_j a_{ij} x_j \qquad \text{(notice } a_{ij} = \frac{\partial x_i'}{\partial x_j}\text{)} \quad .$$

To insure that the primed basis also form an orthogonal set, it is necessary and sufficient to require that the matrix of transformation coefficients obey the relation (cf. equation 4-15 of Goldstein's Classical Mechanics)

$$\sum_{i=1}^{N} a_{i\ell} a_{im} \begin{array}{ll} = 1 & \text{if } \ell = m \\ = 0 & \quad \ell \neq m \end{array} \quad .$$

We are now ready to define the quantities of physics by means of their transformation properties, in terms of the basis transformation coefficients. Let S denote some scalar quantity, V denote some vector quantity, T denote some tensor quantity, and let the argument q represent all variables which are connected to the quantity through the laws of physics. Primes denote quantities measured in the transformed system and unprimes quantities measured in the untransformed system. We now define three types of physical quantites:

Scalar transformation $\qquad S'(q') = S(q)$

Vector transformation $\qquad V_i'(q') = \sum a_{i\ell} V_\ell(q)$

Tensor transformation $\qquad T_{ij}'(q') = \sum_\ell \sum_m a_{i\ell} a_{jm} T_{\ell m}(q)$

With these definitions it can be seen that a quantity can be a scalar under one particular type of transformation, and at the same time be a vector under a different type of transformation. We can also see that many of the quantities which we have naively thought of as scalars, such as energy, temperature (which is really a mean kinetic energy), and time, remain scalar under only the most trivial types of transformations. Similarly the cross-product is often referred to as a vector, which it most certainly is not; it is an antisymmetric tensor, under transformations which invert the coordinates.

We will now attempt to formulate these quantities in a notation which is convenient. Schemes of index notation vary somewhat both from author to author and from problem to problem, but we have found the following scheme to be extremely flexible and concise in a large variety of classes of problems.

## B. Rules of the Game

Rule #1 Denote a component of an indexed quantity by a roman letter index. Treat all quantities in terms of one of its components.

Rule #2 Denote the summation of a quantity over an index by a greek letter dummy index (thus suppressing the summation sign). Such a dummy index can appear upon only one side of any equation, so it can be arbitrarily named and renamed.

Rule #3 Define the Kronecker Delta Tensor:

$$\delta_{ij} \begin{aligned} &= 1 \quad \text{if } i = j \\ &= 0 \quad \text{if } i \neq j \end{aligned}$$

Rule #4 Define the Levi-Civita Tensor Density

$$\varepsilon_{ijk} \begin{aligned} &= 1 \quad \text{if ijk is an even permutation of 123} \\ &= -1 \quad \text{if ijk is an odd permutation of 123} \\ &= 0 \quad \text{if any two (or all three) of the indices ijk are equal.} \end{aligned}$$

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Useful Identities:

The Kronecker Delta, summed over its indices, is equal to the dimensionality of the space

$\delta_{\alpha\alpha}$ = N, the number of coordinates spanning the space

Products of Levi-Civita Tensors can be simplified using the relation

$$\varepsilon_{ijk}\,\varepsilon_{i\ell m} = \delta_{j\ell}\,\delta_{km} - \delta_{jm}\,\delta_{k\ell}$$

which can be verified by careful inspection.

## C. Formulation of Vector Analysis in This Notation

Dot Product of two vectors:

$$\vec{A}\cdot\vec{B} \equiv A_\alpha B_\alpha$$

Cross Product of two vectors:

$$(\overrightarrow{AxB})_i \equiv \varepsilon_{i\alpha\beta} A_\alpha B_\beta$$

Gradient of a scalar:

$$(\vec{\nabla}S)_i \equiv \frac{\partial S}{\partial x_i}$$

Divergence of a vector:

$$\vec{\nabla}\cdot\vec{V} \equiv \frac{\partial V_\alpha}{\partial x_\alpha}$$

Curl of a Vector:

$$(\overrightarrow{\nabla x V})_i \equiv \varepsilon_{i\alpha\beta} \frac{\partial V_\beta}{\partial x_\alpha}$$

## D. Exercises

1. Show that $\delta_{ij}' = a_{i\alpha} a_{j\beta} \delta_{\alpha\beta}$

   and therefore $\delta_{ij}$ does indeed obey the tensor transformation law.

2. Show that, for a coordinate transformation of the form

$$x_i' = a_{i\alpha} x_\alpha$$

   with a corresponding inverse transformation

$$x_j = (a^{-1})_{j\beta} x_\beta'$$

   the inverse transformation matrix is given by

$$(a^{-1})_{\ell m} = a_{m\ell}$$

3. Show that

$$\frac{\partial S'}{\partial x_i'} = a_{i\alpha} \frac{\partial S}{\partial x_\alpha}$$

   so we see that the gradient transforms like a vector. Show also that the Laplacian obeys the transformation

$$\frac{\partial^2 S'}{\partial x_\alpha' \partial x_\alpha'} = \frac{\partial^2 S}{\partial x_\beta \partial x_\beta}$$

   so is therefore a scalar.

## E. Proof of Various Vector Identities by Index Notation

1. $\vec{\nabla} \cdot (S\vec{V}) = S(\vec{\nabla} \cdot \vec{V}) + (\vec{\nabla} S) \cdot \vec{V}$

$$\frac{\partial}{\partial x_\alpha}(S V_\alpha) = S \frac{\partial V_\alpha}{\partial x_\alpha} + \frac{\partial S}{\partial x_\alpha} V_\alpha$$

2. $\vec{\nabla} \times (S\vec{V}) = S(\vec{\nabla} \times \vec{V}) + (\vec{\nabla} S) \times \vec{V}$

$$\varepsilon_{i\alpha\beta} \frac{\partial}{\partial x_\alpha}(S V_\beta) = S \varepsilon_{i\alpha\beta} \frac{\partial V_\beta}{\partial x_\alpha} + \varepsilon_{i\alpha\beta} \frac{\partial S}{\partial x_\alpha} V_\beta$$

3. $\vec{\nabla} \cdot (\vec{U} \times \vec{V}) = \vec{V} \cdot (\vec{\nabla} \times \vec{U}) - \vec{U} \cdot (\vec{\nabla} \times \vec{V})$

$$\frac{\partial}{\partial x_\alpha}(\varepsilon_{\alpha\beta\gamma} U_\beta V_\gamma) = \varepsilon_{\alpha\beta\gamma} V_\gamma \frac{\partial U_\beta}{\partial x_\alpha} + \varepsilon_{\alpha\beta\gamma} U_\beta \frac{\partial V_\gamma}{\partial x_\alpha}$$

$$= V_\gamma \varepsilon_{\gamma\alpha\beta} \frac{\partial U_\beta}{\partial x_\alpha} - U_\beta \varepsilon_{\beta\alpha\gamma} \frac{\partial V_\gamma}{\partial x_\alpha}$$

4. $\vec{\nabla}\times(\vec{U}\times\vec{V}) = \vec{V}\cdot(\vec{\nabla}\vec{U}) - \vec{U}\cdot(\vec{\nabla}\vec{V}) + \vec{U}(\vec{\nabla}\cdot\vec{V}) - \vec{V}(\vec{\nabla}\cdot\vec{U})$

Take the ith component

$$\varepsilon_{i\alpha\beta}\frac{\partial}{\partial x_\alpha}\left[\varepsilon_{\beta\mu\nu}U_\mu V_\nu\right] = \varepsilon_{\beta i\alpha}\varepsilon_{\beta\mu\nu}\left[\frac{\partial U_\mu}{\partial x_\alpha}V_\nu + U_\mu\frac{\partial V_\nu}{\partial x_i}\right]$$

$$= (\delta_{i\mu}\delta_{\alpha\nu} - \delta_{i\nu}\delta_{\alpha\mu})\left[\frac{\partial U_\mu}{\partial x_\alpha}V_\nu + U_\mu\frac{\partial V_\nu}{\partial x_\alpha}\right]$$

$$= \frac{\partial U_i}{\partial x_\alpha}V_\alpha + U_i\frac{\partial V_\alpha}{\partial x_\alpha} - \frac{\partial U_\alpha}{\partial x_\alpha}V_i - U_\alpha\frac{\partial V_i}{\partial x_\alpha}$$

5. $\vec{\nabla}\times(\vec{\nabla}\times\vec{V}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{V}) - \nabla^2 V$

Take the ith component

$$\varepsilon_{i\alpha\beta}\frac{\partial}{\partial x_\alpha}\left[\varepsilon_{\beta\mu\nu}\frac{\partial}{\partial x_\mu}V_\nu\right] = \varepsilon_{\beta i\alpha}\varepsilon_{\beta\mu\nu}\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\mu}V_\nu$$

$$= (\delta_{i\mu}\delta_{\alpha\nu} - \delta_{i\nu}\delta_{\alpha\mu})\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\mu}V_\nu$$

$$= \frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_i}V_\alpha - \frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\alpha}V_i$$

6. $\vec{\nabla}\times(\vec{\nabla}S) = 0$

$$\varepsilon_{i\alpha\beta}\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\alpha}S = -\varepsilon_{i\beta\gamma}\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\beta}S$$

Since α and β are both dummy indices, we may rename α as β and β as α without changing the result.

$\varepsilon_{i\alpha\beta}\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\beta}S = -\varepsilon_{i\alpha\beta}\frac{\partial}{\partial x_\beta}\frac{\partial}{\partial x_\alpha}S = 0$ if $\frac{\partial}{\partial x_\alpha}$ and $\frac{\partial}{\partial x_\beta}$ commute

7. $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{V}) = 0$

$$\frac{\partial}{\partial x_\alpha}\varepsilon_{\alpha\beta\gamma}\frac{\partial}{\partial x_\beta}V_\gamma = -\frac{\partial}{\partial x_\alpha}\varepsilon_{\beta\alpha\gamma}\frac{\partial}{\partial x_\beta}V_\gamma$$

Renaming $\alpha$ and $\beta$ as in 6.

$\frac{\partial}{\partial x_\alpha}\varepsilon_{\alpha\beta\gamma}\frac{\partial}{\partial x_\beta}V_\gamma = -\frac{\partial}{\partial x_\beta}\varepsilon_{\alpha\beta\gamma}\frac{\partial}{\partial x_\alpha}V_\gamma = 0$ if $\frac{\partial}{\partial x_\alpha}$ and $\frac{\partial}{\partial x_\beta}$ commute

## 8. The Radius Vector

We can define a radius vector for an arbitrary number of dimensions, N. (For ordinary xyz coordinate space, N is 3)

$$r^2 \equiv x_\alpha x_\alpha$$

Notice that if we operate with $\partial/\partial x_i$:

$$2r\frac{\partial r}{\partial x_i} = 2x_\alpha \frac{\partial x_\alpha}{\partial x_i} = 2x_\alpha \delta_{i\alpha} = 2x_i$$

Therefore

$$(\vec{\nabla} r)_i \equiv \frac{\partial r}{\partial x_i} = \frac{x_i}{r}$$

Notice also that

$$\vec{\nabla}\cdot\vec{r} = \frac{\partial x_\alpha}{\partial x_\alpha} = \delta_{\alpha\alpha} = N$$

$$(\vec{\nabla}\times\vec{r})_i = \varepsilon_{i\alpha\beta}\frac{\partial x_\beta}{\partial x_\alpha} = \varepsilon_{i\alpha\beta}\delta_{\alpha\beta} = 0$$

We also consider the derivatives of the inverse, 1/r

$$\left(\vec{\nabla}\frac{1}{r}\right)_i \equiv \frac{\partial}{\partial x_i}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{\partial r}{\partial x_i} = -\frac{x_i}{r^3}$$

$$\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}\left(\frac{1}{r}\right) = \frac{\partial}{\partial x_j}\left(-\frac{x_i}{r^3}\right) = \frac{3x_i\frac{\partial r}{\partial x_j}}{r^4} - \frac{\frac{\partial x_i}{\partial x_j}}{r^3}$$

$$= \frac{3x_i x_j - \delta_{ij} r^2}{r^5}$$

## F. Proof of Vector Integral Theorems

DIVERGENCE (Gauss) THEOREM

This theorem relates the integral of a vector divergence over a VOLUME to the integral of the vector over a SURFACE inclosing this volume.

$$\iiint (\vec{\nabla}\cdot\vec{F})\,dV = \iint (\vec{F}\cdot\vec{n})\,dS$$

The proof becomes trivial if we write the left hand side in index notation

$$\iiint \left(\frac{\partial F_\alpha}{\partial x_\alpha}\right)\left|\frac{\varepsilon_{\alpha\beta\gamma}}{2}\right| dx_\alpha\,dx_\beta\,dx_\gamma = \iint \underbrace{\left\{\int \frac{\partial F_\alpha}{\partial x_\alpha}\,dx_\alpha\right\}}_{\int dF_\alpha}\left|\frac{\varepsilon_{\alpha\beta\gamma}}{2}\right| dx_\beta\,dx_\gamma$$

$$= \iint F_\alpha \left|\frac{\varepsilon_{\alpha\beta\gamma}}{2}\right| dx_\beta\,dx_\gamma$$

STOKES THEOREM

This theorem relates the integral of a vector curl over an AREA to the line integral of the vector around the PERIMETER of the area.

$$\iint (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,dA = \int \vec{F}\cdot d\vec{r}$$

Writing the LHS in index notation

$$\iint \left(\left|\frac{\varepsilon_{\alpha\beta\gamma}}{2}\right| \frac{\partial F_\gamma}{\partial x_\beta}\right) dx_\beta\,dx_\alpha = \int \left|\frac{\varepsilon_{\alpha\beta\gamma}}{2}\right| \underbrace{\left\{\int \frac{\partial F_\gamma}{\partial x_\beta}\,dx_\beta\right\}}_{\int dF_\gamma} dx_\gamma$$

$$= \int F_\gamma\,dx_\gamma$$