Quantum statistics: Is there an effective fermion repulsion or boson attraction?

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Physicists often claim that there is an effective repulsion between fermions, implied by the Pauli principle, and a corresponding effective attraction between bosons. We examine the origins and validity of such exchange force ideas and the areas where they are highly misleading. We propose that explanations of quantum statistics should avoid the idea of an effective force completely, and replace it with more appropriate physical insights, some of which are suggested here. © *2003 American Association of Physics Teachers.*

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I. INTRODUCTION

The Pauli principle states that no two fermions can have the same quantum numbers. The origin of this law is the required antisymmetry of the multi-fermion wavefunction. Most physicists have heard or read a shorthand way of expressing the Pauli principle, which says something analogous to fermions being ''antisocial'' and bosons ''gregarious.'' Often this intuitive approach involves the statement that there is an effective repulsion between two fermions, sometimes called an ''exchange force,'' that keeps them spatially separated. We inquire into the validity of this heuristic point of view and find that the suggestion of an effective repulsion between fermions or an attraction between bosons is actually a dangerous concept, especially for beginning students, because it often leads to an inaccurate physical interpretation and sometimes to incorrect results. We argue that the effective interaction interpretation of the Pauli principle (or Bose principle) should almost always be replaced by an alternate physical interpretation that better reveals the true physics.

Physics comes in two parts: the precise mathematical formulation of the laws, and the conceptual interpretation of the mathematics. David Layzer has said,[1] ''There is a peculiar synergy between mathematics and ordinary language ... The two modes of discourse (words and symbols) stimulate and reinforce one another. Without adequate verbal support, the formulas and diagrams tend to lose their meaning; without formulas and diagrams, the words and phrases refuse to take on new meanings.'' Interpreting the meaning of wavefunction symmetry or antisymmetry in a simple pedagogical representation is thus vitally important. However, if those words actually convey the wrong meaning of the mathematics, they must be replaced by more accurate words. We believe that this is the situation with the heuristic ''effective repulsion'' for fermions or ''effective attraction'' for bosons, or ''exchange force'' generally.

We can demonstrate there is no real force due to Fermi/Bose symmetries by examining a time-dependent wave packet for two noninteracting spinless fermions. Consider the antisymmetric wave function for one-dimensional Gaussian wave packets, each satisfying the Schrödinger equation, and moving toward each other:

$$\begin{aligned}\psi(x_1,x_2,t)=C\{&f(x_1,x_2)\exp[-\alpha(x_1-vt+a)^2\\&-\beta(x_2+vt-a)^2]-f(x_2,x_1)\\&\times\exp[-\alpha(x_2-vt+a)^2-\beta(x_1+vt-a)^2]\},\end{aligned}\tag{1}$$

where x_1 and x_2 are the particle coordinates, $f(x_1,x_2)=\exp[imv(x_1-x_2)/\hbar]$, C is a time-dependent factor, and the packet width parameters α and β are unequal. In reality, each single-particle packet will spread with time, but we assume that the spreading is negligible over the short time that we consider the system. At $t=0$, the α-packet is peaked at $-a$ and moving to the right with velocity v, while the β-packet is peaked at $+a$ and traveling to the left with the same velocity. Of course, we cannot identify which particle is in which packet because they are indistinguishable, and each has a probability of being in each packet. At $t=0$ the packets are assumed well separated with little overlap.

At $t=a/v$, the wave function becomes

$$\begin{aligned}\psi(x_1,x_2,t)=C\{&f(x_1,x_2)\exp[-\alpha(x_1)^2-\beta(x_2)^2]\\&-f(x_2,x_1)\exp[-\alpha(x_2)^2-\beta(x_1)^2]\},\end{aligned}\tag{2}$$

and the direct and exchange parts have maximal overlap. The wave function clearly vanishes at $x_1=x_2$ (at all times). At the time $t=2a/v$, the packets have passed through one another and overlap very little again:

$$\begin{aligned}\psi(x_1,x_2,t)=C\{&f(x_1,x_2)\exp[-\alpha(x_1-a)^2-\beta(x_2+a)^2]\\&-f(x_2,x_1)\exp[-\alpha(x_2-a)^2-\beta(x_1+a)^2]\}.\end{aligned}\tag{3}$$

Now the α-packet is peaked at $+a$, but still moving to the right and the β-packet is peaked at $-a$ and still moving to the left. The packets have moved through one another unimpeded because, after all, they represent free-particle wave functions. Describing this process in terms of effective forces would imply the presence of scattering and acceleration, which do not occur here, and would be highly misleading.

Nonetheless, the concept of effective fermion repulsion is evident in many texts, particularly in discussions of the behavior of an ideal fermion gas, a case we explore further in Sec. II. A common usage of the repulsion idea is in the interpretation of the second virial coefficient of an ideal gas. The first correction to the pressure of a classical ideal gas due only to statistics is positive for spinless fermions and negative for spinless bosons. Heer[2] (similar to most other texts that treat the subject, including one authored by one of us[3]) says, ''The quantum correction that is introduced by statistics appears as an attractive potential for Bose–Einstein (BE) statistics and as a repulsive potential for Fermi–Dirac (FD) statistics.'' Pathria[4] carries the idea further, developing a mathematical expression for the effective interaction between fermions or between bosons. Huang[5] also quotes this quantity. This expression first appeared in 1932 in an article by Uhlenbeck and Gropper,[6] who may well be the originators of the whole statistical interaction picture. We discuss this formula in more detail in Sec. II.

Wannier[7] is a bit stronger in his assessment of the quantum thermal distribution function for fermions: ''The particles exert a very strong influence on each other because a particle occupying a state excludes the others from it. This is equivalent to a strong repulsive force comparable to the strongest forces occurring in the problem.'' Leighton[8] omits the word ''effective'' in discussing the so-called fermion interaction: ''As compared with the behavior of hypothetical but distinguishable particles, Bose particles exhibit an additional attraction for one another and tend to be found near one another in space; Fermi particles, on the contrary, repel one another and tend not to be found near one another in space.''

Griffiths[9] has done an interesting calculation of the average distance between two particles at positions x_1 and x_2 when one is in state ψ_a and the other in ψ_b; the two functions are orthogonal and normalized. For distinguishable particles with wave function $\psi_a(x_1)\psi_b(x_2)$, the mean-square separation is

$$\langle (x_1-x_2)^2\rangle_{dist}=\langle x^2\rangle_a+\langle x^2\rangle_b-2\langle x\rangle_a\langle x\rangle_b\,, \tag{4}$$

where $\langle x\rangle_i=\int dx\, x|\psi_i(x)|^2$. For spinless fermions the wave function must be antisymmetrized, and for bosons symmetrized, giving

$$\Psi=\frac{1}{\sqrt{2}}[\psi_a(x_1)\psi_b(x_2)\pm\psi_a(x_2)\psi_b(x_1)], \tag{5}$$

where the upper sign is for bosons and the lower for fermions. From this form it is easy to compute the corresponding mean-square separation as

$$\langle (x_1-x_2)^2\rangle_\pm=\langle (x_1-x_2)^2\rangle_d\mp 2|\langle x^2\rangle_{ab}|, \tag{6}$$

where $\langle x\rangle_{ab}=\int dx\, x\psi_a^*(x)\psi_b(x)$. Thus he finds that bosons tend to be closer together and fermions farther apart when compared to distinguishable particles. Griffiths comments that, ''The system behaves as though there were a 'force of attraction' between identical bosons, pulling them closer together, and a 'force of repulsion' between identical fermions, pushing them apart. We call it an exchange force, although it's not really a force at all—no physical agency is pushing on the particles; rather it is a purely a geometrical consequence of the symmetrization requirement.'' This wording shows more care than the works cited above and is thus less likely to be misinterpreted. However, the term ''force'' has explicit meaning for physicists. It implies a push or pull, along with its associated acceleration, deflection, scattering, etc. Are these elements properly associated with the exchange force? If not, then the term should be replaced by words that convey more accurate connotations.

Our intention is not to be critical of authors for using the words ''repulsion'' and ''attraction'' in describing the statistical effects of wavefunction antisymmetry or symmetry. This concept has been with physics since the early days of quantum mechanics. Nevertheless, it is important to examine the usefulness of this heuristic interpretation of the mathematics. As Layzer has pointed out,[1] no such interpretation can carry the whole weight of the rigorous mathematical formulation; however, if a heuristic interpretation brings along the baggage of subsequent misconceptions, then physicists must be more circumspect in its use.

For example, consider the following case where there is a complete breakdown of the concept. Suppose two spinless fermions or bosons have a completely repulsive interparticle potential and impinge on one another at energies low enough that there is only s-wave scattering. As we show in Sec. III, if the scattering amplitude for distinguishable particles is f, then the scattering amplitude for fermions vanishes identically, whereas it is $2f$ for bosons. In this case the statistical symmetry has diminished the interaction for fermions—not made it more repulsive—and it has enhanced the interaction for bosons—not made it less repulsive.

Wherever the idea of an effective force breaks down (as it does in our wave-packet description and in the s-wave scattering example), we need to replace this interpretation with other heuristic interpretations that better represent the physics. This is our aim in the examples we analyze below.

In Sec. II we examine more closely the physics that gives rise to the idea of an effective statistical interaction between quantum particles and derive the Uhlenbeck–Gropper formula for the interaction. Section III will take the opposite point of view, and present cases where the idea is highly misleading and where the effect is actually opposite the usual implication. Section IV summarizes our conclusions.

II. EXAMPLES OF THE STATISTICAL INTERACTION

There are several contexts where the idea of a statistical interaction arises naturally, and seems to imply an effective force. The virial correction to the pressure of an ideal gas is most likely the origin of this idea of effective interaction. The physics of white dwarf stars is another classic example of ''Fermi repulsion.'' The diatomic hydrogen atom is bound in the electron singlet state, while the triplet is unbound, which is often used as an example of the effective repulsion between like-spin electrons due to the Paul principle. When two rare gas atoms approach one another, there is an exponential repulsion between the atoms, which often is explained by the electron statistical repulsion. Similarly, when trapped bosons condense, they collapse to a smaller region in the center of the trap, which gives the impression of an effective boson statistical attraction. In each of these cases we will show that relying on the intuitive idea of Pauli repulsion or Bose attraction may hinder understanding of the basic phenomena. Alternative explanations are provided.

Virial expansion: A real gas has an equation of state that differs from that of an ideal classical gas. For high temperature T and low density n of the gas, the pressure P can be written

$$P=nk_BT(1+nB(T)), \tag{7}$$

where k_B is Boltzmann's constant and B is the second virial coefficient. Equation (7) gives the lowest terms of the virial expansion, a series in powers of $n\lambda^3$, where λ is the thermal wavelength, given by $\lambda = \sqrt{h^2/(2\pi m k_B T)}$ for particles of mass m.

For ideal spinless fermions and bosons, standard calculations give the effect of Fermi or Bose symmetry:[10]

$$B(T) = -\eta \frac{\lambda^3}{2^{5/2}}, \tag{8}$$

where $\eta = \pm 1$, with the plus sign for bosons and the minus for fermions. Thus fermions exert a larger pressure and bosons a smaller pressure on the walls than a classical gas at the same temperature.

Compare this result with that for a classical interacting gas, where the second virial coefficient is given by[10]

$$B(T) = \frac{1}{2} \int d\mathbf{r}\,(1 - e^{-\beta U(\mathbf{r})}), \tag{9}$$

where $U(r)$ is the real interatomic potential at separation r and $\beta = 1/k_B T$. It is evident from Eq. (9) that a completely repulsive potential leads to a positive $B(T)$ and a positive contribution to the pressure, while an attractive one results in a negative contribution.

A connection to the Fermi or Bose ideal gas is made by considering the pair density matrix given by

$$\begin{aligned} G(1,2) &= V^2 \frac{\langle \mathbf{r}_1 \mathbf{r}_2 | e^{-\beta H_{12}} | \mathbf{r}_1 \mathbf{r}_2 \rangle}{\mathrm{Tr}(e^{-\beta H_{12}})} \\ &= \lambda^6 \sum_{\mathbf{p}_1 \mathbf{p}_2} \psi_{\mathbf{p}_1}(\mathbf{r}_1) \psi_{\mathbf{p}_2}(\mathbf{r}_2) e^{-\beta(\epsilon_{p_1} + \epsilon_{p_2})} \\ &\quad \times (1 + \eta P_{12}) \psi^*_{\mathbf{p}_1}(\mathbf{r}_1) \psi^*_{\mathbf{p}_2}(\mathbf{r}_2), \end{aligned} \tag{10}$$

where $\psi_{\mathbf{p}_i}(\mathbf{r}_i)$ is a plane-wave momentum state for particle i and P_{12} is the permutation operator interchanging $\mathbf{r}_1$ and $\mathbf{r}_2$. The single-particle energy is $\epsilon_p = p^2/2m$. If we change the momentum sums to integrals and carry out the calculations, we obtain the following result, which depends on relative position r_{12} only:

$$G(r_{12}) = (1 + \eta e^{-2\pi r_{12}^2/\lambda^2}). \tag{11}$$

The purely classical ideal gas result would correspond to $\eta = 0$ with no correlation between particles. Fermions, on the other hand, have G small within a thermal wavelength, an example of the spatial consequences of the Pauli principle. Bosons have G larger than the classical value. This result is consistent with the Griffiths' calculation of $\langle (x_1 - x_2)^2 \rangle$ cited in Sec. I.

Spatial correlations in a dilute classical gas are described by the two-particle distribution function given by $G_{\mathrm{cl}}(1,2) = e^{-\beta U(\mathbf{r})}$. Thus, as in Refs. 4–6, we can identify an effective statistical potential by analogy as

$$U_{\mathrm{eff}}(r) = -k_B T \ln(1 + \eta e^{-2\pi r_{12}^2/\lambda^2}). \tag{12}$$

This quantity is plotted in Fig. 1; it is purely repulsive for fermions and attractive for bosons. If we substitute Eq. (12) into the classical expression for the second virial coefficient, we obtain precisely the result in Eq. (8). A repulsive potential excludes atoms from approaching too closely and raises the pressure; fermions also have an ‘‘excluded volume’’ of λ^3 and an increase in pressure. This comparison seems to be the major impetus behind the concept of effective force as applied to Fermi statistics. Is the physics similar enough for the analogy to be useful? Our opinion is that it is not very helpful, as we argue below.

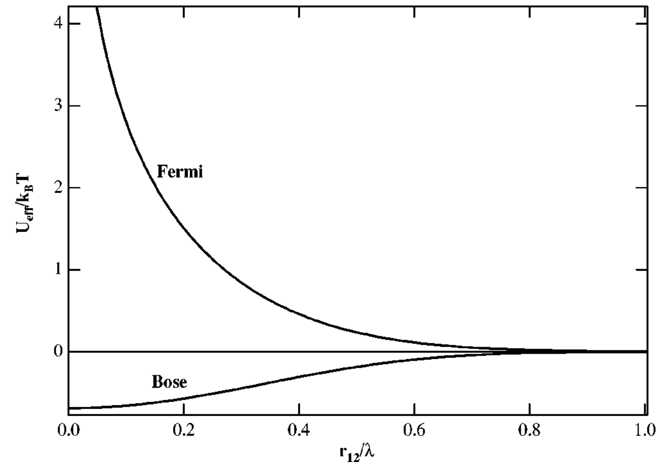

Fig. 1. Plot of the effective statistical interaction versus position. For bosons this function is attractive; for fermions it is repulsive.

In a classical gas the rms average momentum remains $\sqrt{\overline{p^2}} = \sqrt{3mk_BT}$ even when there are interactions. Pressure is force per unit area and the force comes from the impulse of an atom striking the wall. The average force that one single particle in a vessel exerts on the wall is, by the impulse-momentum theorem, $F = \Delta p/\Delta t$, where Δp is twice the average momentum and Δt is the average time over which the force is exerted. Here Δt is *not* the time of contact, but rather the time for an atom to cross the width L of the container, that is, $\Delta t = mL/\overline{p}$. When we make the volume of an ideal classical gas smaller (at constant T), $\overline{p}$ is unchanged, but the transit time Δt is diminished causing the pressure to increase. Analogously, if we turn on the repulsive interactions in a classical gas with no change in the temperature or $\overline{p}$, the pressure rises because of a decreased average transit time: some molecules bounce off others back to the wall they just left. But this is *not* what happens in the fermion case.

The idea that the correlation hole in the two-body density Eq. (11) gives rise to ‘‘bounces’’ or deflections of fermions from one another is a misconception that arises from the idea of a Pauli repulsion. When we compare Fermi gas dynamics to that of classical statistics, what is altered is not the effective L in the transit time, but rather the $\overline{p}$ in both Δp and in Δt. For a given value of T, the momentum distribution in an ideal Bose or Fermi gas differs from that in an ideal classical gas. The exact quantum second virial coefficient is given by[10]

$$B(T) = \frac{1}{2V} \int d\mathbf{r}_1\, d\mathbf{r}_2 [1 - G(1,2)]. \tag{13}$$

This result explains why the substitution of U_{eff} into the classical equation gives the exact answer. Nevertheless, it is not the spatial dependence of G that gives us physical insight; it is the *momentum* dependence. If we carry out the position integration indicated in Eq. (13) with G as given by Eq. (10), the result is

$$\frac{1}{2V}\int d\mathbf{r}_1\,d\mathbf{r}_2\,G(1,2)$$

$$=\frac{\lambda^6}{V}\left\{\frac{1}{2}\left[\sum_{\mathbf{p}_1\mathbf{p}_2} e^{-\beta(\epsilon_{p_1}+\epsilon_{p_2})}+\eta\sum_{\mathbf{p}} e^{-2\beta\epsilon_p}\right]\right\}. \tag{14}$$

The quantity inside the curly brackets is the partition function for two quantum particles. The first term is the classical partition function, and its contribution already is accounted for in the classical ideal gas pressure; it cancels out in Eq. (13). The second term corrects the incorrect classical momentum distribution represented by the first term. The classical term includes double-occupation states; for fermions the second term cancels these. For bosons, the classical counting undercounts these double-occupation terms, and the second term corrects that fault as well. Writing the second virial coefficient in momentum space clarifies how the change in momentum distribution affects the pressure. For bosons, there is a lowering of the average momentum so the force on the wall is lessened. For fermions, the momentum is raised increasing the pressure. The idea of an effective repulsion between fermions ignores the real physics and gives a very poor analogy with classical repulsive gases.

White dwarf stars and related objects: It is the fermion zero-point pressure that prevents the collapse under gravitational forces of the white dwarf star. Krane[11] says, ‘‘A white dwarf star is prevented from collapse by the Pauli principle, which prevents the electron wave functions from being squeezed too close together ... Will the repulsion of the electron wave functions due to the Pauli principle be able to prevent the collapse of any star, no matter how massive?’’ (This line of reasoning leads to a discussion of neutron stars.) We believe this qualitative picture of what goes on in a white dwarf star could, as with the second virial coefficient interpretation, be greatly improved by a discussion in terms of the momentum-space features of the Pauli principle. Most elementary discussions of white dwarfs[11,12] incorporate a discussion of Fermi repulsion by doing a dimensional analysis that equates the zero-point energy of the ideal Fermi gas to the gravitational self-energy of the star matter. The Fermi temperature is much greater than the physical temperature in the star so that the $T=0$ fermion gas is used as a model.

An alternative physical description arises from considering the hydrostatic equilibrium conditions of the star.[13] The star is assumed to contain N nuclei (assumed to be all helium) in radius R. A spherical shell of thickness dr at radius r has an outward force due to the difference between the pressure $P(r)$ on the inner surface and the pressure $P(r+dr)=P+dP$ (with $dP<0$) on the outer surface, caused by the nonuniform nuclear number density of the star, $n(r)$. This net outward force $4\pi r^2\,dP$ is balanced by the gravitational pull toward the center due to the total mass $M(r)$ enclosed by the shell. The mass of the shell itself is $4\pi r^2 n(r)dr\,m_{\mathrm{He}}$, where m_{He} is the helium mass, so that

$$dP=-\frac{GM(r)n(r)dr\,m_{\mathrm{He}}}{r^2}. \tag{15}$$

The crucial idea is that P is the pressure of a degenerate electron gas with the electron density maintained by charge neutrality at twice the helium number density $n_e(r)=2n(r)$. For a nonrelativistic model the Pauli pressure at $T=0$ is given by standard statistical arguments[12] as $P\approx\hbar^2 n_e^{5/3}/m_e$. Chandrasekhar[13] develops a second-order differential equation for $n(r)$ from these steps. We can do a simple dimensional analysis based on Eq. (15) by replacing dP/dr by $-P/R$, $n(r)$ by N/R^3, $M(r)$ by $M(R)=m_{\mathrm{He}}N$, etc., to arrive at

$$R=\frac{\hbar^2}{Gm_e m_{\mathrm{He}}^2}\frac{1}{N^{1/3}}\approx\frac{1}{M^{1/3}}. \tag{16}$$

Equation (16) is the usual nonrelativistic result, which does not demonstrate the collapse at some large M like the relativistic case, but gives the idea behind the stability of the star.

The gravitational attraction on a mass element is balanced by the difference in Pauli pressure across the mass shell. In order to develop a qualitative argument for the strong density dependence of the Pauli pressure that supports the star against gravitational collapse, we can return to the argument used for the virial coefficient. In a box of side L, the pressure is force per unit area A, or $P=(N/A)\Delta p/\Delta t$. But the average momentum per particle Δp imparted to the wall for a degenerate Fermi gas is of order p_{F}, the Fermi momentum. The transit time is $\Delta t\sim Lm_e/p_{\mathrm{F}}$, so that

$$P\approx\frac{N}{AL}\frac{p_{\mathrm{F}}}{m_e/p_{\mathrm{F}}}=n_e\frac{p_{\mathrm{F}}^2}{m_e}. \tag{17}$$

The Fermi momentum is strongly dependent on the density because of the necessity of filling the single-particle energy levels with two per momentum state. This requirement is $N=(2V/h^3)\int d\mathbf{p}\,n_{\mathbf{p}}$, with $n_{\mathbf{p}}$ a step function cutting off at $p=p_{\mathrm{F}}$. This integral gives $p_{\mathrm{F}}=\hbar(3\pi^2 n_e)^{1/3}$. Note that p_{F} is related to a deBroglie wavelength by

$$p_{\mathrm{F}}=\frac{\hbar}{\lambda}\approx\hbar n_e^{1/3}. \tag{18}$$

Thus the maximum wavelength is approximately the interparticle separation, which one can argue is necessitated by the Pauli principle requiring that the electrons be in single-particle wave packets compact enough that they do not overlap. This argument is about quantum mechanical wave function correlation rather than an argument based on an effective force. The connection to the Pauli pressure is the high momentum that this correlation induces. We end up with

$$P\approx n_e\frac{p_{\mathrm{F}}^2}{m_e}\approx\frac{\hbar^2}{m_e}n_e^{5/3}. \tag{19}$$

If by ‘‘preventing the wave functions from being squeezed too close together,’’ [11] we mean that the fermion wave function must have sufficient curvature for nodes to appear whenever any two coordinates are equal, then the idea leads directly to the correct behavior. This extra curvature requires higher Fourier components. The pressure differs from one kind of statistics to another directly because of differing momentum distributions; the Fermi distribution involves larger average momenta, giving it a Pauli pressure. The idea of ‘‘wave function repulsion’’ as a correlation that leads to this momentum distribution might be useful, although the word ‘‘repulsion’’ still carries the connotation of a force, which is less useful.

The physical explanations of neutron stars,[14] strange quark matter,[15] and the Thomas–Fermi model of the atom,[16] are analogous to the white dwarf star in that the Pauli pressure of a Fermi fluid is the basis of resistance to compression.

The hydrogen molecule and interatomic forces: The singlet electron state of hydrogen is bound while the triplet state is unbound. Is it a case of the Pauli repulsion giving the spatially antisymmetric state associated with the triplet higher energy? Griffiths,[9] applying the discussion of exchange forces to this problem, says "The system behaves as though there were a 'force of attraction' between identical bosons, pulling them closer together... If electrons were bosons, the symmetrization requirement... would tend to concentrate the electrons toward the middle, between the two protons..., and the resulting accumulation of negative charge would attract the protons inward, accounting for the covalent bond... But wait. We have been ignoring spin..." He then writes about the fact that the entire spin and space wave function must be antisymmetric and obtains the proper bonding in the singlet state. He shows that for the spatially antisymmetric triplet state "the concentration of negative charge should actually be shifted to the wings..., tearing the molecule apart."

Although this explanation is very carefully worded and provides a very useful physical picture of the hydrogen bond, a strikingly different picture of covalent bonding and antibonding is given by the work of Herring.[17] Herring argues that the energy difference between singlet and triplet states (in widely separated atoms at least) is properly interpreted as a splitting between atomic levels due to tunneling. Consider the hypothetical case of two spinless, distinguishable electrons in a hydrogen molecule. The Hamiltonian has the form

$$H = t_1 + t_2 + V(12) + U(1) + U(2), \tag{20}$$

in which t_i is the kinetic energy operator for particle i, $V(12)$ represents the particle–particle interaction, and $U(i)$ is an external double-well potential representing the attraction of the ith electron to the two nuclei located, say, at R_a and R_b. The Hamiltonian is symmetric under interchange of the two particles, so the eigenfunctions must be either symmetric or antisymmetric, even for these distinguishable particles. Let ψ_+ and ψ_- represent the lowest symmetric and antisymmetric eigenfunctions, respectively, with corresponding energies E_+ and E_-.

The combination,

$$\phi_{ab}(1,2) = \frac{1}{\sqrt{2}}(\psi_+ + \psi_-), \tag{21}$$

is a function for which particle 1 is localized near site R_a and particle 2 near site R_b. If P_{12} is the permutation operator, then the function $P_{12}\ \phi_{ab}(1,2) = \phi_{ab}(2,1) = \phi_{ba}(1,2)$ is localized about the exchanged sites, that is, particle 1 is localized near site R_b and particle 2 near site R_a. Herring calls the functions $\phi_{ab}(1,2)$ and $\phi_{ba}(1,2)$ "home-base functions." If one sets the initial conditions such that the particles are in $\phi_{ab}(1,2)$, then the two particles will tunnel through the double-well barrier between $\phi_{ab}(1,2)$ and $\phi_{ba}(1,2)$ with frequency $\omega = (E_+ - E_-)/\hbar$. We can write the energy of these two lowest states for distinguishable particles as

$$E_\pm = E_0 \pm J, \tag{22}$$

where $E_0 = (E_+ + E_-)/2$ and $J = (E_+ - E_-)/2$. A theorem[18] states that the symmetric nodeless state must be the ground state, thereby implying that J is negative. This energy splitting occurs independent of any spin effects.

If the two particles are spin-1/2 fermions, the same physics holds, except now the symmetric wave function ψ_+ must be associated with an antisymmetric spin function while ψ_- must be associated with a symmetric spin function in order to keep the entire wave function overall antisymmetric. The result is that Eq. (22) is replaced by

$$E_\pm = E_0 \pm J\sigma_1 \cdot \sigma_2, \tag{23}$$

where σ_i is a Pauli spin matrix. This operator expression, the second term of which is often called the "exchange interaction," acts in spin space to associate the correct spin state (singlet or triplet) with the correct energy.[19–22] Because the symmetric spatial state is the ground state, as in the distinguishable particle case, the singlet state energy of the electrons is lower than that of the triplet state.

Note that we are not criticizing the idea quoted from Ref. 9 that the lowering of the energy in the spin singlet state can be associated with the concentration of the electron cloud in the region between the two nuclei. However, the energy lowering would arise even if the two particles were distinguishable; so it does not actually stem from their fermion character.[23,24] Of course, the fact that the corresponding spin state must be singlet is a fermion property. The suggestion that Fermi statistics or Pauli repulsion plays a role in the lowering of the singlet relative to the triplet state of H_2 misses the essential fact that much of the energy difference is due to the splitting between tunneling states and that the tunneling ground state must be nodeless and symmetric.

Let's continue this discussion, but with the hydrogen nuclei replaced with helium nuclei. We can get an idea of the behavior of the electronic energy for this pair of helium atoms by using the same symmetric and antisymmetric wave functions. Because of the Pauli principle, the two extra electrons would (in some approximation) be placed in the spatially antisymmetric, antibonding, triplet state, thereby losing the tunneling energy advantage of the symmetric state. This extra energy supplies a physical explanation for the repulsive interatomic interaction when the closed-shell electron clouds start to overlap. Within the Born–Oppenheimer approximation[25] the electronic energy (plus the internuclear Coulomb repulsion) is used as a potential energy for the atomic nuclei. The short-ranged repulsive part of this interaction potential between two rare-gas atoms is often described by a phenomenological $1/r^{12}$ or exponential repulsion.[26,27] This case is an example of a real repulsion arising; however, it is a repulsion between the nuclei and not the electrons.

Although the Pauli principle is certainly vital in understanding molecular forces, the idea of an effective fermion statistical repulsion has never really entered the picture. Indeed, we believe its introduction short-circuits the discussion and could cause one to miss the basic physics.

Bose–Einstein condensation: Condensation of bosons into a harmonic trap might seem the best example of boson effective attraction.[28] The condensate in a trap is a noticeably smaller object than the cloud of noncondensed atoms surrounding it. Of course, the real reason for this is that the ground state in the trap of the interacting bosons has a smaller radius than that of the excited states. The particles are correlated to be in the same state; in this case it is a spatially more compact one.

One of the present authors has also used the following argument[3] to explain the fact that the lowest excited mode in a Bose fluid is a phonon rather than a single-particle motion: "Bosons prefer to be in the same state with one another, so that if one atom is pushed on by an external force, all the particles within a deBroglie wavelength λ (which is large at low temperature) want to move in the same way. The collective motion of a sound wave allows this while the single-particle motions are frozen out by this tendency." This argument might seem to be based on the idea of a kind of boson effective attraction. A more rigorous argument is given by Feynman.[29] If Φ is the ground state of the Bose fluid, then one might suppose that $e^{i\mathbf{k}\cdot\mathbf{r}_1}\Phi$ is an excited state involving a single particle with momentum $\mathbf{k}$. However, the state has to be symmetric, so this state must be replaced by $\Sigma_i e^{i\mathbf{k}\cdot\mathbf{r}_i}\Phi$, which is precisely the one-phonon state. The particles "preferring to be in the same state" is a verbal expression to represent wave-function symmetry. Superfluids can be described by a "wave function" (order parameter), which depends on a single position variable, has a magnitude and phase, and represents the superfluid distribution. It costs energy to make this function nonuniform, as when a vortex is present. The system "prefers" to have the same phase and amplitude throughout, a property sometimes called "coherence." Any idea of an effective boson attraction is better replaced by this latter concept.

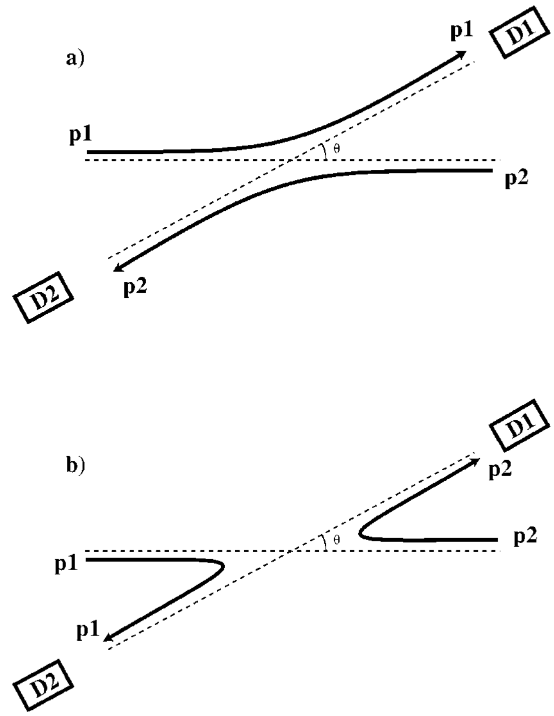

Fig. 2. Diagrams for two-particle scattering.

III. WHERE THE IDEA OF A STATISTICAL INTERACTION FAILS

We have argued that the idea of a statistical fermion repulsion or boson attraction might lead one to miss the essential physics of the physical effect being explained. Worse however, is the fact that this idea might cause misconceptions and lead to incorrect conclusions. We present here some cases where that might occur.

The other spin state: Most of the textbooks quoted in Sec. I say unequivocally that fermions repel and bosons attract without the qualification of, say, the term "spinless." These books have ignored, at some pedagogic risk, the effects of spin, which is usually taken into account only later. The effective repulsion or attraction (if there were one) is an effect of the spatial part of the wave function only. If the total spin state is symmetric, the space wave function is antisymmetric for fermions and symmetric for bosons, leading to the effects envisioned in most textbooks. However, when the total spin state is antisymmetric (as for two spin 1/2 particles in a spin singlet, or for two spin 1 particles in the $S=1$, $m_s=0$ state) the roles of fermions and bosons are reversed. Two spin 1/2 fermions in the spin singlet state behave like two spinless bosons, and two spin 1 bosons in the $S=1$, $m_s=0$ state behave like spinless fermions.

Scattering theory: When two particles scatter elastically via a repulsive force, the idea of an additional effective interaction due to Fermi or Bose symmetry can lead to trouble. In the center-of-mass frame the two particles approach from opposite directions and scatter into opposite directions as shown in Fig. 2(a). If the particles are distinguishable, the probability of detecting particle p_1 in detector D_1 and particle p_2 in detector D_2 is given by

$$P(p_1 \text{ in } D_1 \text{ and } p_2 \text{ in } D_2)=|f(\theta)|^2, \tag{24}$$

where $f(\theta)$ is the scattering amplitude.[30] Similarly, the probability of detecting particle p_1 in detector D_2 and particle p_2 in D_1 [as in Fig. 2(b)] is given by

$$P(p_1 \text{ in } D_2 \text{ and } p_2 \text{ in } D_1)=|f(\pi-\theta)|^2. \tag{25}$$

When we do not care which particle goes to which detector, but just want to measure the cross section for either particle in a detector, the probability for a particle in detector D_1 is

$$P(p_1 \text{ or } p_2 \text{ in } D_1)=|f(\theta)|^2+|f(\pi-\theta)|^2. \tag{26}$$

Because the particles are in principle distinguishable, there is no interference between amplitudes, even if the detectors themselves do not identify the difference between particles.

Now suppose the two particles are indistinguishable. In this case the two amplitudes corresponding to Figs. 2(a) and 2(b) interfere, and must be combined before squaring. If the particles are identical fermions, the two-particle wave function is antisymmetric with respect to particle exchange. Because Figs. 2(a) and 2(b) are related by the exchange of the two particles in the final state, they contribute to the total amplitude with opposite signs. Thus, the probability to detect a fermion in detector D_1 is

$$P_{\text{Fermi}}(p \text{ in } D_1)=|f(\theta)-f(\pi-\theta)|^2. \tag{27}$$

Equation (27) is obviously different from the distinguishable case in Eq. (26). The difference is especially remarkable at $\theta=\pi/2$, where the fermion scattering probability vanishes. Moreover, in the limit of s-wave scattering, which is a good approximation for some low energy cases, the scattering is independent of the angle θ and the fermion probability for scattering is zero for all angles. A similar argument holds for bosons, but with the amplitudes adding instead of subtracting, leading to the scattering probability:

$$P_{\text{Bose}}(p \text{ in } D_1)=|f(\theta)+f(\pi-\theta)|^2. \tag{28}$$

In this case, the scattering probability at $\theta = \pi/2$ is twice the value for distinguishable particles. For s-wave scattering P_{Bose} is a factor of 2 times the distinguishable value at all angles.

The interpretation of these results in the context of an effective fermion repulsion or an effective boson attraction is quite confusing. For scattering at 90 degrees, or for s-wave scattering at all angles, it looks as if the total repulsive force is reduced for fermions (leading to a smaller scattering probability) and enhanced for bosons (leading to a larger scattering probability). This scattering probability result contradicts the notion that the scattering force should be supplemented by an effective repulsion for fermions and partially canceled by an effective attraction for bosons. It clearly demonstrates why the idea of an effective repulsion or attraction is a dangerous concept.

Focusing on the direct effects of the Bose or Fermi symmetry leads to a more useful conceptual approach to scattering. For two identical particles, the total spin state is symmetric. For fermions having a total spin state that is symmetric (either both spins up or both spins down), the spatial part of the wave function itself must be antisymmetric as in Eq. (5). For this wave function, the amplitude for the two fermions to be in the same place ($r_1 = r_2$) is obviously zero. As noted in Sec. I, two identical fermions are on average farther apart than two distinguishable particles would be under the same circumstances. Consequently, the fermions interact *less* and are less likely to scatter. One can similarly argue that bosons are closer together on average, interact more and are *more* likely to scatter.

This conclusion is true whether the scattering force is repulsive or attractive, but it depends critically on the spin state of the two particles. For identical particles the spin state is necessarily symmetric, forcing the fermion spatial wave function to be antisymmetric or the boson wave function to be symmetric. However, if the two particles are in an antisymmetric spin state, for example, two fermions in a spin-zero state, the conditions are reversed.

As a specific example of repulsive scattering, consider the quantum electrodynamic interaction of two electrons (Møller scattering). In QED, there are two lowest-order Feynman diagrams that contribute to the scattering amplitude with opposite signs, corresponding to the direct and exchange diagrams of Fig. 2. In the nonrelativistic limit, the electron spins do not change as a result of the interaction, due to the fact that the low energy interaction occurs primarily via an electric field. There is then only one final spin state to consider when doing the calculation. If the initial state has both particles with spin up, then the final state also has both spins up. This case is treated in introductory particle physics texts,[31] and the cross section for scattering is

$$\frac{d\sigma}{d\theta}(\text{identical spins}) = \frac{m^2\alpha^2}{32p^4}\left(\frac{1}{\sin^2\frac{\theta}{2}} - \frac{1}{\cos^2\frac{\theta}{2}}\right)^2, \quad (29)$$

where α is the fine structure constant and p is the momentum of the electrons in the center-of-mass frame. We note that the cross section at $\theta = \pi/2$ vanishes, as it should.

To explore the case of an antisymmetric spin wave function, we can also calculate the scattering cross section for electrons in a spin-zero state:

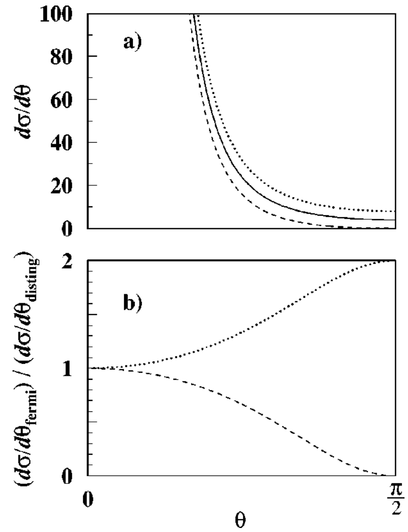

Fig. 3. (a) The cross section for electron scattering as a function of scattering angle for identical spins (dotted line), for the spin-zero state (dashed line), and for the hypothetical case of distinguishable particles (solid line). All cross sections are in units of $m^2\alpha^2/32p^4$. (b) The differential cross section $d\sigma/d\theta$ for identical (dotted line) and spin zero (dashed line) electrons relative to $d\sigma/d\theta$ for distinguishable electrons.

$$\frac{d\sigma}{d\theta}(\text{spin zero}) = \frac{m^2\alpha^2}{32p^4}\left(\frac{1}{\sin^2\frac{\theta}{2}} + \frac{1}{\cos^2\frac{\theta}{2}}\right)^2. \quad (30)$$

Equation (30) differs from the identical spin case, Eq. (29), in the relative sign of the two terms.

These results should be compared to what the cross section would be if the two electrons were distinguishable. In that case, only the direct diagram of Fig. 2 contributes, and the cross section for scattering with both spins up turns out to be the same as the spin-averaged cross section for electron–muon scattering found in many texts, with the muon mass set equal to the electron mass. After symmetrizing about $\theta = \pi/2$ to account for detectors that are sensitive to either particle, the cross section can be written as

$$\frac{d\sigma}{d\theta}(\text{distinguishable electrons}) = \frac{m^2\alpha^2}{32p^4}\left(\frac{1}{\sin^4\frac{\theta}{2}} + \frac{1}{\cos^4\frac{\theta}{2}}\right). \quad (31)$$

The three cases are plotted as a function of scattering angle in Fig. 3(a). All cross sections are symmetric about $\pi/2$, so only the range 0 to $\pi/2$ is plotted. As expected, the symmetric spin case gives the smallest scattering cross section, the antisymmetric spin case gives the largest cross section, and the case of distinguishable particles is in between. Moreover, as Fig. 3(b) shows, the ratios of the cross sections change as a function of scattering angle. At small scattering angles, the fermion cross sections are almost the same as the distinguishable particle cross section. The maximum difference occurs at $\theta = \pi/2$. It is difficult for any kind of effective fermion interaction to capture this effect, and moreover, the idea of an effective Fermi repulsion gives the wrong sign in the case of a repulsive scattering force.

Transport theory: Consider the example of thermal conductivity in a polarized fermion gas. One might think from the idea of Pauli repulsion that increasing polarization would shorten the particle's mean-free path in the gas, which in turn would lower the thermal conductivity κ. The opposite behavior is more likely to happen. At sufficiently low temperature where s-wave scattering predominates, polarization will actually cause a dramatic increase in κ, because, as we have just seen, the s-wave scattering cross section between like-spin fermions vanishes and only scattering between unlike spins, which now happens less often, can contribute to the mean free path.

We treat a gas obeying Boltzmann statistics, but having full quantum-mechanical collisions. For this treatment to be applicable, the deBroglie wavelength must be larger than the scattering length, but smaller than the average separation between particles. If the temperature is low enough, s-wave scattering will predominate. This situation can occur, for example, in trapped Fermi or Bose gases. The heat current for spin species μ in the temperature gradient dT/dz is given by arguments analogous to those for an unpolarized gas:[32]

$$J_\mu = -n_\mu \bar{v} l_\mu k_B \frac{dT}{dz}, \tag{32}$$

where n_μ is the density of μ spins, $\bar{v}$ is the average velocity of either spin species, l_μ is the mean free path of a μ spin, and k_B is the specific heat per molecule. In Eq. (32) μ is $+$ for up spins and $-$ for down spins; there is a separate heat equation for each spin species. We have dropped any constant factors in the expression. When s-wave scattering dominates, up spins can interact only with down spins and not with each other, and vice versa. Thus the mean free path is $l_\mu = \bar{v}\tau_\mu$ where τ_μ is the inverse of the scattering rate given by

$$\frac{1}{\tau_\mu} = n_{-\mu} \bar{v} \sigma_{+-}, \tag{33}$$

with σ_{+-} the cross section for spin up–down scattering. The spin density $n_{-\mu}$ occurs on the right in Eq. (33) because it is that of the target particles for the incoming μ spins. The result is that

$$J_\mu = -\frac{n_\mu}{n_{-\mu}} \frac{\bar{v} k_B}{\sigma_{+-}} \frac{dT}{dz}. \tag{34}$$

If $n_+/n_- \gg 1$, the heat current J_- for down spins is negligible compared to J_+ for the up spins, and the thermal conductivity is

$$\kappa = \frac{n_+}{n_-} \frac{\bar{v} k_B}{\sigma_{+-}}. \tag{35}$$

For high polarizations $(n_+/n_- \gg 1)$ κ can be very large. The increase in κ and other transport coefficients for polarized systems has been predicted theoretically[33–35] and also observed experimentally.[36] A similar increase also occurs if the particles are degenerate. The idea of a statistical repulsion is counterintuitive to this result.

Ferromagnetism: A very simple mean-field picture of magnetic fluids is provided by a model in which the particles interact by s-wave scattering only.[12] Thus again there is no up–up or down–down interactions, and the energy of the system of N_+ up spins and N_- down spins is given by

$$E = E_+ + E_- + gN_+N_-, \tag{36}$$

where E_σ is the total kinetic energy of the σ spins, which is proportional to $N_\sigma^{5/3}$ as in the ideal Fermi gas. The interaction parameter g measures the potential energy between up and down spins. Two up spins (or two down spins) do not "see" each other in this model. If $g > 0$, this model has three possible states. If g is small, then the system favors less kinetic energy by having $N_+ = N_- = N/2$. That is, the system is antiferromagnetic. However, if g is large enough, then either all the spins are up or all down to minimize the potential energy and ferromagnetism results. The kinetic energy in this case is larger than it would be if $N_\sigma = N/2$, but the potential energy is zero. The wave function antisymmetry between two like spins has made them invisible to one another and noninteracting, because their minimum separation is greater than the range of the interaction. The idea of a Pauli exchange force would lead one to assume a higher associated potential energy [as in Eq. (12)], but what actually happens is that the kinetic energy is raised by having more like spins, while at the same time statistics favors lowering the potential energy.

IV. DISCUSSION

Our goal in this paper has been to clarify the idea of the statistical effects sometimes referred to as "exchange forces." We believe the term "force" used in this context may mislead students (and even more advanced workers), who might misinterpret the geometrical effect being described. We have given several examples of instances where statistical or exchange forces have been invoked to provide a conceptual explanation of the physics. We have not introduced any new physics in these examples, but we have tried to show how a teacher or writer might provide an alternative interpretation that avoids the exchange force terminology and thereby arrives at a deeper heuristic understanding of the physics. Indeed we identified several cases where the concept of an effective statistical force could lead to the opposite of the correct answer. When a concept has that potential, it is time to replace it.

Our alternative heuristics have taken several forms. We like Griffiths'[9] wording: "it is not really a force at all... it is purely a geometric consequence of the symmetrization requirement." For same-spin fermions, the requirement that the wave function vanish whenever two particles are at the same position means that the wave function must have increased curvature, which leads to an enhanced momentum distribution. Indeed in many cases, the real statistical effect corresponds to a change in kinetic energy (that is, the momentum distribution) as in the explanation of the virial pressure or the white dwarf star, whereas a force picture leads to a change in potential energy as in the Uhlenbeck–Gropper potential of Eq. (12). Equally helpfully, the geometrical interpretation leads directly to the changes in the average particle separation as compared to distinguishable particles, with same-spin fermions farther apart on average, which nicely explains the scattering results for which same-spin fermions have a reduced interaction.

It is difficult to overestimate the importance of the the conceptual element of physics. Introductory courses have been constructed that leave out much of the mathematics and concentrate only on the "conceptual" side of the subject.[37] Moreover, we emphasize to our students that they have not understood a theory until they can describe the physics in simple conceptual terms. Given that emphasis, we offer the following guiding principle regarding statistical symmetries: "May the force be *not* with you."

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