Concept of the exponential law prior to 1900

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(Received 5 October 1977; accepted 29 March 1978)

The historical development of a quantitative perception of the processes of exponential growth and decay is traced from its ancient origins through pre-20th century mathematical formulations and physical applications. Evidence indicates that the concepts of the mean life and the half life and their relationships to linear differential rate equations and to multiple simultaneous and sequential processes, as well as the nonlinear conditions which bring about the breakdown of the exponential law, were all understood and presented in written records long before their now nearly synonymous application to nuclear radioactivity. Exponential processes were among the earliest quantitative concepts to be mathematically formulated, and our modern understanding of them can be enhanced by historical perspective.

I. INTRODUCTION

During this century studies of dynamical processes which have a fixed halving or doubling interval have provided a wealth of information concerning the nature of our physical world. For example, nuclear decay processes have been used to ascertain the dates of cosmological, geological, and archaeological events, as well as to provide a sensitive probe of the properties of the nucleus itself. Similarly, atomic and molecular decay processes have been used to deduce stellar and interstellar chemical abundances, and play a key role in the understanding of luminescent phenomena, laser technology, chemical reactions, etc. Many other practical examples of exponential and nearly exponential processes exist,[1] which make them extremely useful pedagogically. Students with limited mathematical background often find the exponential, with its infinite number of nonvanishing differential rates of change, to be less formidable than a few-times differentiable quantity such as uniformly accelerated speed or position. A general familiarity with the exponential concept exists today, which is at least partially due to the wide interest in radioactivity, and indeed the exponential law is almost always discussed by analogy with this application. However, the radioactive decay law was not discovered until 1900,[2] when exponential processes had already been studied, utilized, and described in written records for a very long time. Since it involved a newly discovered phenomenon, the early literature of radioactivity often presented the various physical and mathematical properties of the exponential process as though they also were new. Thus the definitions of the half life and mean life, the solutions to the coupled rate equations and their implications for complex systems containing simultaneous and sequential processes,[3] and other similar features were deduced from first principles, without reference to earlier applications in other fields. It is therefore interesting to trace the historical development of the concepts of exponential growth and decay prior to its popularization in radioactivity, for it was one of the earliest dynamical processes to be formulated mathematically, and its basic simplicity may be partially concealed by the modern mathematical symbolism in which it has been embedded.

II. ANCIENT FORMULATIONS

A. The geometric progression

The modern algebraic exponential concepts and notations were not developed until the 17th century. However, physical examples of exponential growth and decay appeal to very basic number concepts and could be recognized with little mathematical abstraction. The exponential law is most concretely stated as a one-to-one correspondence between an arithmetic progression (formed by repeated successive addition of a quantity) such as

$$0, 1, 2, 3, 4, \ldots$$

and a geometric progression (formed by repeated successive multiplication by a quantity) such as

$$1, 2, 4, 8, 16, \ldots$$

or its reciprocal, the halving progression. Here the exponential law is to be distinguished from the power law, in which there is a one-to-one correspondence between an arithmetic progression and a progression such as 1, 4, 9, 16, . . . , formed when terms in an arithmetic progression are raised to a fixed power.

One of the most striking features of a geometric progression is its generation of large and small numbers. For example, 80 successive doublings of a 1-cm length exceeds the distance to the Andromeda nebula, and 80 successive halvings of 1 mole of material splits the last remaining molecule of the sample. In the 3rd century B.C. Archimedes[4–6] compared large numbers with the elements of a geometric progression generated by repeated successive multiplication with the factor 10^8 and classified their sizes according to the elements of a corresponding arithmetic progression. However, many other features of the geometric progression had been recognized long before the time of Archimedes, and had, e.g., been recorded on Egyptian papyrus[7] and Sumerian cuneiform tablets.[8]

B. Summation properties

The ancient Egyptians were masters of the geometric progression, and constructed their entire system of basic arithmetic operations around it.[9] Thus multiplication was effected by first forming a geometric progression by successive doubling of the multiplicand, then adding together selected elements from this progression so as to correspond to the multiplier. Although they used a decimal number system for integers, a binary system was used for subdivision of units, and noninteger quantities were often expressed as a sum of selected elements from the so-called "Horus Eye" fractions $1/2$, $1/4$, . . . , $1/64$. Through this repeated manipu-

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Table I. The 7 house-cat problem from the Rhind Egyptian papyrus of around 1650 B.C.

Multiplication by 7	Houses	Cats	Mice	Spelt	Hekats	Total
House inventory	*1*	*7*	*49*	*343*	*2 401*	2 801
Doubled	*2*	*14*	*98*	*686*	*4 802*	5 602
Doubled again	*4*	*28*	*196*	*1372*	*9 604*	11 204
Village inventory	7	49	343	2401	16 807	19 607

lation, various properties of the geometric progression became known, and examples of their insights can be seen on papyrus records. Problem number 79 of the Rhind papyrus, compiled by the scribe A 'h-mosè in about 1650 B.C., is the famous puzzle of the seven housecats[10] described in Table I. The entries in roman type appeared on the papyrus and have been interpreted to mean "In a certain village there were 7 houses; each house had 7 cats; each cat caught 7 mice; each mouse would (were it not for the cats) have eaten 7 ears of spelt; each ear of spelt produced 7 hekats (about half a peck) of grain at harvest. How many hekats of grain were saved by the presence of the cats and (unasked, but answered) how many heterogeneous elements are there in the sequence?" The probable method of solution was suggested by Neugebauer[11] in 1926, and is indicated by the entries in italics. The village inventory for each category is obtained by doubling the house inventory twice and adding the three entries. The village inventory for one item is numerically equal to the house inventory for the next item, so the process can be continued until the village inventory of saved grain (16 807 hekats) is obtained. The heterogenous sum (19 607) is computed by two methods which provide a check: by summing the five items in the village inventory, and by taking the house total, doubling it twice, and adding these three numbers. The entire procedure was performed using only simple addition, by virtue of the properties of the geometric progression.

This particular problem of sevens has propagated throughout the ages, and occurs in the writings of Fibonacci in 1202 (with 6 items: old women, packs, sacks, loaves, knives, and sheaths)[12] and still persists in a Mother Goose rhyme (with 4 items: wives, sacks, cats and kits, all met on the way to St. Ives).[13] These summation properties of a geometric progression can provide insight into what we would now call the integral properties of the exponential function. The fact that the sequence formed by a cumulative sum of the successive elements in a geometric progression grows itself in nearly geometric ratio must have been at least intuitively recognized by the ancient Egyptians.[14] An explicit formula relating the cumulative sum to the extreme elements in the sequence can be deduced by noting that multiplication of the sum by the term ratio merely shifts the elements of the sequence, yielding the same sum when corrected for the extreme elements. This quantitative insight was probably beyond the Egyptian mathematics of the Rhind papyrus,[11] but the general formula was certainly widely known by the end of the 4th century B.C.[15] and was elegantly and generally proven using ratio and proportion in *Euclid's Elements* (Book IX, Proposition 35) in about 300 B.C. In modern notation Euclid's formula is written

$$\sum_{n=0}^{N} r^n = \frac{r^{N+1}-1}{r-1} \qquad (1)$$

Cases for which this sum converges for large N are particularly interesting. It has been reported[16] that one bit of papyrus from around 1800 B.C. contained the phrase "multiply by one-half to infinity," which could suggest that the ancient Egyptians considered the infinite-halving progression, and were perhaps confronted by the concept of a limit. It was this series which Zeno of Elea used to formulate his "Paradox of the Dichotomy" which disturbed Greek mathematics in around 450 B.C. and brought about a concern for rigor in passing to a limit. *Euclid's Elements,* Book X, Proposition 1 deals with the convergence of the dichotomy problem. Questions of convergence were carefully considered by Archimedes when, in around 250 B.C., he summed the infinite quartering series in Proposition 23 of his treatise *Quadrature of the Parabola.*[17]

C. Differential Rate Properties

The connection between rates of change and the geometric progression was also recognized very early in a quite practical application. The practice of paying 10%–20% annual interest on loan of produce or precious metals was common in Babylonia in 2000 B.C.[18] (later interest rates were legislated to a lower level). Quantitative concepts were conveyed through specific numerical (sexagesimal based) examples using precomputed tabulations. According to the practice of the times, compound interest was accrued at an agreed fixed interval, and for an uncompleted interval simple interest, based on the indebtedness at the last completed compounding interval, was assessed. Thus the indebtedness at an arbitrary time could be computed by a linear interpolation between the entries of the compound interest tables (which are of course the elements of a geometric progression).

A Louvre cuneiform tablet[19] from around 2000 B.C. (see Fig. 1) asks the question "How long does it take an amount invested at 20% annually compounded interest to double itself?" It goes on to outline the solution, utilizing a tabulation of the quantity $(6/5)^n$, and linearly interpolating between $n = 3$ and $n = 4$ to obtain the correct answer, 3 yr 9 and 4/9 months. Thus, in some sense, this tablet can be regarded as expressing the essentials to the exponential solution of a differential rate equation and relates a discretely compounding analog of the half-life to the rate constant. It is not recorded whether the Sumerian financiers noted that not only the indebtedness, but also the interest, the interest on the interest, etc., increased at 20%/yr, but such insights were clearly within reach for a profit-minded investor in 2000 B.C.

Two Berlin tablets[20] from the same period also consider a 20% annual interest rate, but with compounding periods only once every 5 yr. Thus the compounding and doubling periods coincided and the indebtedness could be computed by linear interpolation using a tabulation of 2^n. The specific problem provided the initial and final indebtedness and asked for the number of elapsed compounding periods. The method of solution was complicated, but it has been suggested by Neugebauer[20] that it utilized the operational concept of, and gave a symbolic notation for, inverse exponentiation, the equivalent of a logarithm with the base 2. However, due to the linear interpolation, it was at best a logarithmic *characteristic,* lacking a *mantissa* and not used for Napierian multiplication. Lacking was the concept of *continuous* compounding, that is (in modern notation), calculation of a^x when x is not an integer. Tabulations were

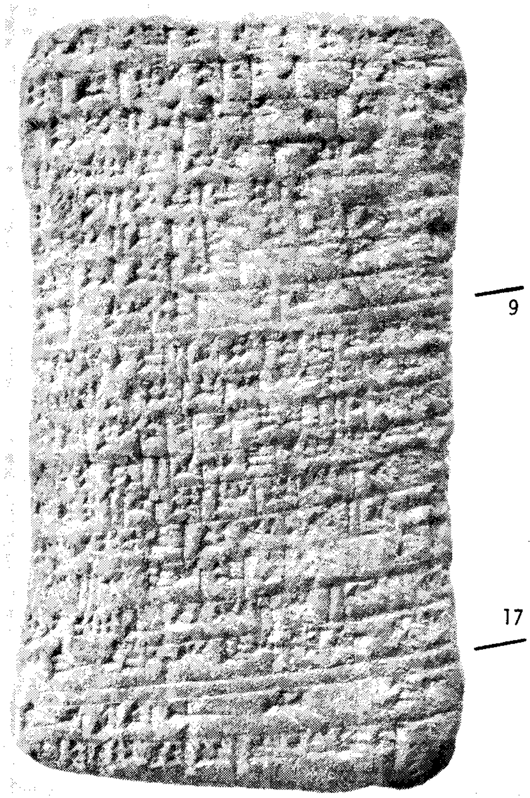

Fig. 1. Louvre cuneiform AO 6770 from about 2000 B.C. On lines 9 and 10 the question is posed "How long does it take for an amount invested at 20% annually compounded interest to double itself?" Lines 11–14 describe a linear interpolation between the 3-yr and 4-yr compounded totals, line 15 gives the number 2 plus 33/60 plus $20/60^2$, and lines 16 and 17 explain that this is the number of months short of 4 yr at which the doubling occurs.

made by successive repeated multiplication, and interpolation methods involving either extraction of roots or infinite series expansion were to come much later.

III. PERCEPTIVE LINEARIZATION OF STIMULI

A. Weber-Fechner law

Although the mathematical properties of exponential processes were clearly understood in the ancient world, many physical examples went unrecognized because of the approximately logarithmic response of human perception, which causes exponentially varying stimuli to seem linearly varying to the senses. This tendency was quantitatively formulated by Weber and Fechner[21] in the mid-19th century. They observed that the ability to discriminate between two similar stimuli depends upon the ratio of (rather than the difference between) the intensities of the stimuli. This is true, for example, for both the intensities and frequencies of light and sound waves. Although only incidental to the development of the exponential concept, two examples of the linear perception of an exponential variation have implications worthy of mention—these are the studies of musical pitch and of stellar magnitudes.

B. Pythagorean harmony

Anticipating the Weber-Fechner law, the Pythagoreans recognized that human perceptions of differences in musical pitch correspond to *ratios* of vibrating string or air column lengths. It is said that Pythagoras himself discovered that pleasing chords are achieved if length ratios correspond to successive elements of an arithmetic progression, 1/2, 2/3, and 3/4, which define, respectively, the octave, fifth, and fourth. The definition of the tone, or difference between a fourth and a fifth, must soon have followed. The Pythagorean diatonic scale was constructed by forming sequential length ratios of either 9/8 (the tone) or 256/243 (the lemma). Aristoxenos broke with the Pythagorean concept of harmony and constructed an even tempered scale using the ability of the ear to distinguish an equal succession of increments of pitch, which results in an exponentially decreasing series of string or air column lengths. He then attempted to develop a mathematical formulation in which musical intervals are calculated by addition rather than multiplication, which should have concluded with the discovery of logarithms. Unfortunately Aristoxenos divided the intervals arithmetically rather than by extraction of roots,[22] despite the fact that these calculations must have differed measurably from the even tempered (exponential) string length progression determined by a good ear. Thus Aristoxenos was only prorating the "interest" between "compounding periods" as had the Sumerian bankers long before, and the resemblance to logarithms was more a coincidence than a conceptual breakthrough.

C. Stellar magnitudes

In a similar way, the difference in perceived brightness between two light sources of the same color distribution is essentially proportional to the ratio of their intensities, another example of the Weber-Fechner Law. In the Almagest, Ptolemy cataloged the apparent brightness of many stars according to six classes. We now know that these classes differ progressively by an intensity ratio of about 2.2 to 2.8, hence the classes are an arithmetic progression describing a crude geometric progression of stellar intensities. During the year 1572 there was a great supernova upon which Tycho Brahe made very careful observations. He recorded the dates at which the brightness of the supernova passed from one of Ptolemy's classes to the next, as judged by adjacent Almagest stars. The supernova intensity remained within a class for essentially a *constant interval,* equal to about 2 months. Thus the decay of the supernova was exponential, and Tycho's sharp eyes and record book served as a direct reading semilogarithmic analyzer. Baade[23] has studied the observations of Tycho Brahe, and supplemented them with the Chinese records of the Crab Nebula supernova of 1054, Johannes Kepler's observations of the 1604 supernova, and his own measurements of the 1938 supernova and has deduced that the light curve of this type of supernova has a half-life of 55 days. This lifetime has been attributed[24] to the decay of the element californium-254, which has been found to be produced in hydrogen-bomb explosions.

IV. BIOLOGICAL GROWTH

Possibly the most important example of exponential growth (and deviations from it) concerns biological growth. An early formulation of this application was made in 1202 by Leonardo of Pisa, called Fibonacci. As a mathematical exercise he asked the question "How many pairs of rabbits can be produced from a single pair in a year if each pair begets a new pair every month, which the second month become productive, and no deaths occur?"[25,26] The result is the familiar "Series of Fibonacci"

$$1, 1, 2, 3, 5, 8, 13, \ldots$$

in which each term after the second is the sum of the previous two. For large numbers the ratio of successive terms increase by a factor which approaches $(\sqrt{5} + 1)/2$ (the "golden section"[27]), and hence it approaches a geometric progression. The Fibonacci numbers occur naturally in, e.g., the arrangement of scales in a fir cone and in the florets of various flowers. This phenomenon was studied by, among others, Leonardo da Vinci, Goethe, and Kepler, and is known as *phyllotaxis*.[28] A rectangle can be constructed by a successively spiraled juxtaposition of squares with sides corresponding to the Fibonacci numbers. For large numbers the locus of outer adjacent corners of this figure approaches a logarithmic spiral, which occurs, e.g., in the Nautilus shell and the Ram's horn.

The implications of this problem were used by Leonhard Euler[29] as calculational exercises in 1735. He showed that six survivors of the Flood would, in 200 yr, have 10^6 descendants if their population grew by about 1/16 per year. In 400 yr this would be 10^{11} descendants which he said would exceed the number which the earth could sustain. He also showed that mankind would double itself in 100 yr if it increased by about 1/144 per year, and that a 1% yearly growth rate would cause the population to increase an order of magnitude every 231 yr. The economic and social consequences of such calculations were beginning to arouse interest, and in 1753 Robert Wallace[30] proposed that mankind naturally increases by successive doubling, and tends to do so thrice in 100 yr. These ideas influenced Thomas Malthus who, in his famous essay[31] of 1798, asserted that the population, if unchecked, tends to increase in geometrical ratio while subsistence increases only arithmetically. This had far-reaching consequences in addition to the economic and social controversies which it sparked. The essay was read by both Charles Darwin and Alfred Russell Wallace, and provided them with the principle of natural selection in the struggle for survival within an otherwise exponentially growing population, and inspired them independently to the theory of evolution. The science of statistics (the word was taken from *staatswissenschaft*) was still new, and Malthus suffered from a lack of statistical material, so his aphorism could not be quantitatively formulated through mathematical models (early population inventories were seldom trustworthy, since they were taken for purposes such as taxation, induction into military service, and forced labor, and it was not in the individual's best interests to be counted, or to give correct information). Malthus also discussed deviations from a purely geometrical progression, citing periodic "sickly" periods which reduced the population, only to be followed by periods of increased fertility. The first mathematically analyzable collection of social data was produced by Adolphe Quetelet[32] in 1835. Quetelet recognized quantifiable obstacles to indefinite growth. His friend Pierre Verhulst[33] discussed a mathematical model in 1845, adding a quadratic retardation term and obtained the s-shaped "logistic curve," which differs from exponential growth unless the linear term dominates. This model will be discussed in more detail in Sec. VII A.

V. THE 17th CENTURY: FORMULATIONS

A. Development of logarithms

The seventeenth century brought forth the symbolic notations and graphical representations which are the framework of modern quantitative thought. However, the development of logarithms occurred prior to and independently of this framework, in purely numerical terms to satisfy practical calculational needs. Thus while the existence of logarithms greatly aided later formulations of the exponential function, the developers of logarithms did not require or even possess the exponential function concept.[34] Construction of a table of logarithms involved computation of very dense sets of paired arithmetical and geometrical progressions and the recognition of their conjugate operations of addition and multiplication. Thus if we express the first logarithms published by John Napier[35] in 1614 in modern symbolic notation, these Napierian logarithms (Nap log) are defined by the relationship

$$x \equiv q(1 - 1/q)^{\text{Nap log } x}, \qquad (2)$$

where Napier chose $q = 10^7$. Our modern natural logarithms (ln) are instead defined

$$x \equiv \lim_{q \text{ large}} (1 + 1/q)^{q \ln x}, \qquad (3)$$

and Napier's logarithms are thus related to modern natural logarithms by

$$\text{Nap log } x = \ln(10^{-7}x)/\ln(1 - 10^{-7}). \qquad (4)$$

Note that Nap log x has *no base* (in the sense $b^{\text{Nap log } x} \neq x$) and the sum of the logarithms of two quantities is *not* equal to the logarithm of the product of the two quantities. Nevertheless, in seeking to expedite his numerical computations, Napier made important innovations which were later utilized in the development of the exponential concept. For example, Napier considered contributions of the order of 1 part in 10^{10} to be negligible for his practical purposes, and thus used the approximation (numerically, not in our modern symbols)

$$10^7(1 - 10^{-7})^{x+100} \simeq (1 - 100 \times 10^{-7})[10^7(1 - 10^{-7})^x] \qquad (5)$$

which is the first term of the binomial expansion. Thus he performed the equivalent of 100 successive multiplications simply by moving the decimal point (which Napier himself invented exactly for this purpose) five places to the left and subtracting. In addition Napier interpolated between integer numbers of successive products, not by extraction of roots but by setting *differences* between elements in the arithmetic progression proportional to *ratios* of the corresponding elements in the geometrical progression. Thus Napier used series expansion techniques, and viewed the operation as a continuous, interpolatable, almost functional process, essential ingredients of later exponential formulations.

Although Napier's first logarithms were not compatible with modern definitions, the second edition of the English translation of his *Descriptio*, made by Edward Wright and published in 1618, contained an anonymous Appendix (probably written by William Oughtred) containing the first table of what we now call natural logarithms.[36] Thus, this marks the beginning of the concept of the quantity we now call (after Euler) e, the base of natural logarithms. An extended table of natural logarithms was published by John Speidell in 1622, but Napier and Henry Briggs had already recognized the practical advantages of a set of logarithms based upon the decimal system. Napier's death in 1617 left the task of calculation to Briggs, who completed it with his *Arithmetica Logarithmica* in 1624. In contrast to the methods of successive *powers* of a number close to unity which were possible with Napier's logarithms, Briggs was forced to begin with the definitions $\log 10 \equiv 1$ and $\log 1 \equiv 0$, and interpolate to find other logarithms by tedious successive extraction of *roots*. Thus the further development of the concept of the number e was delayed until the later introduction of infinite series methods.[37]

B. Symbolic representation

The modern notation of a right superscript index to denote successive multiplication was introduced by René Descartes in his *La Géométrie* in 1637. The French word *exposant* or "index" became synonymous with this particular use of an index, and led to the terminology "exponential." The notation had advantages over those employed earlier by Nicole Oresme (ca. 1323–1385). Nicolas Chuquet (~1484), Michael Stifel (~1544), Simon Stevin (~1585), and others.[38] Although Oresme used fractional powers to denote roots and Chuquet used negative powers to denote reciprocals and zero power to denote unity, the presentation by Descartes used only integer powers. However, literal, fractional, and negative exponents were quickly added to Descartes' notation by Wallis, Newton, and others. Despite the widespread use of this exponential notation which followed, its relationship to the logarithm was not utilized or even clearly recognized until the end of the 17th century.

C. Graphical representation

Thus the development of concepts of the exponential and logarithmic *functions* did not occur as a result of manipulation of this symbolic notation, but involved geometrical considerations of graphical representations of constructed loci. The logarithmic spiral was discussed by Descartes in these terms in a letter to Mersenne in 1638. About the same time Descartes received a letter from a jurist named Florimond de Beaune asking, among other things, for the area under a curve for which the ratio of the ordinate to the subtangent is proportional to the difference between the ordinate and the abscissa.[39] Descartes' answering letter of 1639 indicates that he was aware of some of the properties of what we now call the exponential curve, but he did not name the curve or mention the logarithm. The geometrical properties of the exponential curve were probably first studied by Evangelista Torricelli[40] in around 1644, but the term "exponential" was not associated with this curve until much later. Torricelli proposed two names for this function: the "hemihyperbola" since it resembles a hyperbola but possesses only one asymptote; and the "linea logarithmica sive Neperiana" because it could be constructed using Napier's logarithms. The latter was shortened to "logarithmica" and was the accepted name for the inverse logarithm until well into the 18th century. Some historians have incorrectly translated the word "logarithmica" as "logarithm" rather than "exponential," leading them to underestimate the degree of understanding of these functions in the latter half of the 17th century. Using only classical Greek geometry, Torricelli demonstrated that the subtangent (the ratio of the ordinate to the slope) of this curve is a constant. By approximating the curve by a geometrical sequence of rectangles and using Eq. (1) he showed that the area under the curve between two abscissas is the difference between the ordinates multiplied by the subtangent. Thus he stated in geometrical terms the differential and integral properties of the logarithmica (exponential) function. The emergence of the subtangent as the fundamental parameter describing this curve is particularly noteworthy since we would today recognize it as the distance over which the curve falls to $1/e$ of its original value. Torricelli also noted that the area under the semi-infinite curve is given by the product of the initial ordinate and the subtangent. Thus the subtangent on a time abscissa can be regarded as the average time for the curve to fall to zero, or as we now call it, the mean life. Torricelli died in 1647, leaving this manuscript unpublished [it was published in 1900 (Ref. 40)].

The findings of Torricelli were extended and publicized by Christiaan Huygens during the period 1661–1690.[41] Although he did not specifically cite Torricelli, Huygens adopted his terminology and called the function the "logarithmica" when he wrote in Latin and the "logarithmique" when he wrote in French. In a 1661 manuscript[42] Huygens constructed the logarithmica (see Fig. 2) using a halving geometrical progression interpolated with Briggs' decimal base logarithms,[43] and reexamined its geometrical properties. His demonstration of the constancy of the subtangent is particularly interesting, since it compares the subtangent to the halving interval used in constructing the curve. In units of the decimation interval he computed both the subtangent (by finite differences) and the halving interval. Using 18-place logarithms[43] he found their ratio to be

$$\frac{\text{subtangent}}{\text{halving interval}} = \frac{434\,294\,481\,903\,251\,804}{301\,029\,995\,663\,981\,195}.$$

Thus Huygens' proof contains the ratio of the mean life to the half life (he suggested the rational approximation 13/9).

D. Physical application

Huygens studied the logarithmica as a mathematical exercise in 1661, but in 1668[44] he found an application in the gravitational fall of an object through a medium which exerts a retarding force proportional to the velocity of fall (see Fig. 3). Although Newton's laws of motion and the Newton-Leibniz calculus were not yet formulated, Huygens obtained a geometrical solution by plotting the acceleration versus the time, and computing the former point-by-point by subtracting from g at each time abscissa an amount proportional to the area under the curve up to that point. In modern symbols this involves the solution to the (then unformulated) Newton's law equation

$$a(t) = g - K \int_0^t dt'\, a(t') \tag{6}$$

by the finite sum approximation

$$a_n = g - K\Delta t \sum_{i=0}^{n-1} a_i. \tag{7}$$

Huygens easily recognized the resulting curve as the logarithmica, and expressed the drag coefficient in terms of the subtangent. He then undertook a series of experiments

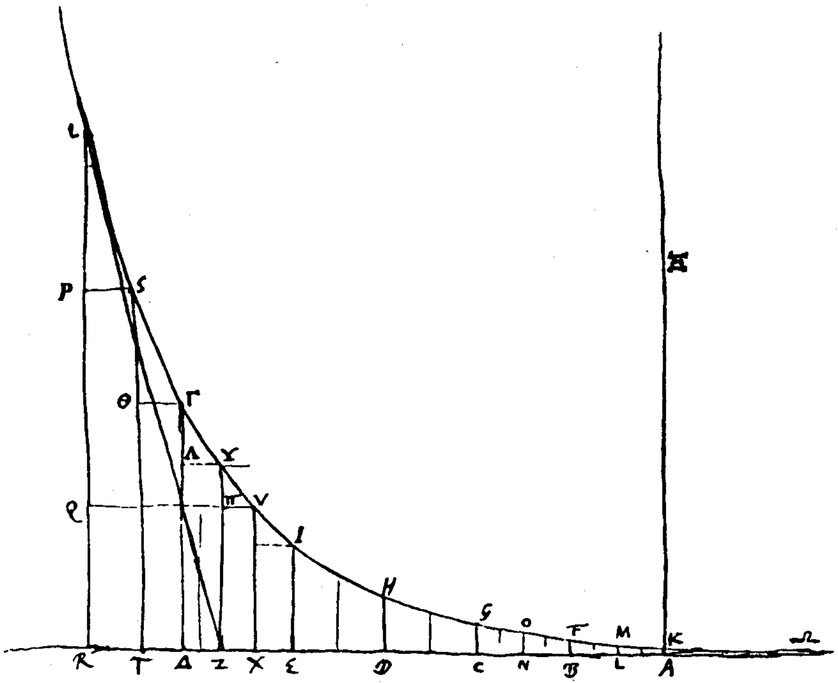

Fig. 2. Construction of the exponential curve from a 1661 manuscript of Christiaan Huygens. The equal segments AB, BC, CD, etc., were set along the abscissa and a doubling progression, AK, BF, CG, etc., was erected thereupon. Huygens used this diagram to study the subtangent and the quadrature of the curve.

to verify the results. Unfortunately, for the situations he measured he found that a velocity-squared dependence was a better model. In 1669,[45] he attempted to repeat his geometrical solution with a velocity-squared retarding force, but found that the result was not as simple (the curve becomes a squared hyperbolic secant) and did not pursue it further except to report his findings on the logarithmica and viscous ballistics before a meeting of the Paris academy in 1669, and also as an Appendix[46] to his *Traité de la Lumière,* published in 1690. Thus the first printed account of the logarithmic curve was probably that of James Gregory,[47] published in 1667. The work of Gregory was possibly not independent of that of Torricelli, since Gregory had spent the previous five years in Italy in close contact with Torricelli's pupil Stefano degli Angeli.[48,49] In 1701 Guido Grandi[50] demonstrated in a more rigorous way the theorems enunciated by Torricelli and Huygens.

E. Infinite series representation

The geometrical studies of Torricelli and Huygens demonstrated many of the properties of a curve whose ordinates are given by the logarithmica of the abscissa, and the computations of Napier and Briggs provided a numerical relationship between a given number and its logarithm. (Despite a tendency of historians to refer to it as the "logarithmic curve," early graphical studies usually chose the asymptote as the axis of the independent variable, as a function of which it was properly an exponential or logarithmica curve.) However, an analytical relationship by which one could compute the ordinate given an arbitrary abscissa or vice versa was still lacking. This was provided by the development of an infinite series representation first for the logarithm and then for its inverse, during the period 1664–1670.

Infinite series methods were being developed independently by several workers during this period, with the earliest efforts toward logarithms and their inverses probably being due to Isaac Newton. In 1664–65 Newton generalized the binomial series to include negative, noninteger, and literal powers, which he used to generate infinite series representations for computations of roots, inverses, and quadratures of (areas under) curves. Among the first examples which he considered was the quadrature of the hyperbola $y = 1/(1 + x)$. This he binomial-expanded and integrated the resulting power series term by term, using the already known formula for the quadrature of a power curve. Thus he obtained the area z as[51]

$$z = x - x^2/2 + x^3/3 - x^4/4 + x^5/5 - \cdots . \quad (8)$$

Gregory St. Vincent had already shown in 1647 that area elements under this hyperbola between successive abscissas are equal if the abscissa interval increases in a geometrical progression, and his pupil Alfons Anton de Sarasa had in 1649 interpreted this result to connect the hyperbola with the logarithm of the abscissa.[52,53] Newton realized this[54] and used $z = \ln(1 + x)$ to compute a number of logarithms from Eq. (8).[55] However, these calculations lay unpublished among his notes until 1669, when he hurriedly assembled them into his manuscript[56] "De Analysi per Aequationes Infinitas." The reason for publication at this time was the publication by Nicolaus Mercator (not the map maker) of the booklet *Logarithmotechnia* in 1668. When Newton received the booklet in September 1668 he was shocked[57] to find that it contained his reduction of $\ln(1 + x)$ to an infinite series (albeit by long division rather than by the binomial expansion). Newton then hastened to prepare his "De Analysi" which he communicated to Isaac Barrow in July 1669. The manuscript not only contains Eq. (8), but its inverse, achieved by neglecting terms higher than x^5 and solving the resulting quintic equation to obtain

$$x = z + z^2/2 + z^3/6 + z^4/24 + z^5/120 + \cdots . \quad (9)$$

Newton correctly asserted that the denominator is a factorial, thus generalizing his quintic solution. Although it established Newton's priority in many infinite series expansions, the manuscript was not widely circulated, and was not actually published until 1711. However, word of these series methods spread to Leibniz, who requested informa-

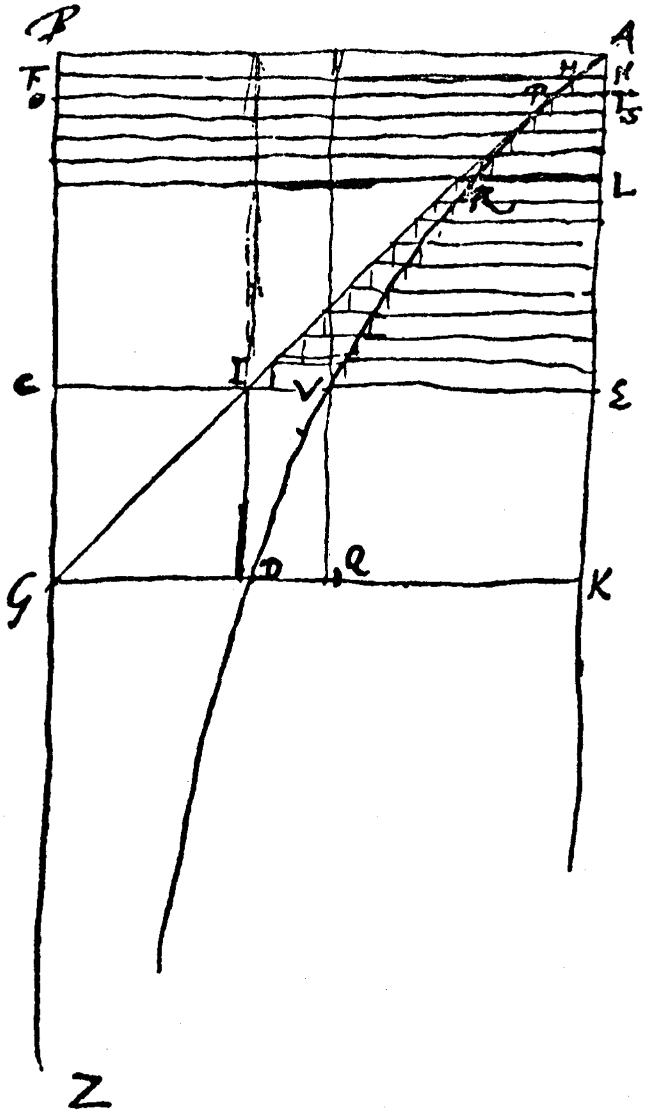

Fig. 3. Geometrical formulation of the acceleration of a body which falls under the combined influence of gravity and a velocity proportional retarding force, from a 1668 manuscript of Christiaan Huygens. Huygens computed the curve by subtracting from g at each point an amount proportional to the area under the curve up to that point, and recognized the resulting curve as the "logarithmica" (exponential) which he had studied earlier.

tion through Henry Oldenburg, then secretary of the Royal Society. This resulted in two letters[58] on this subject from Newton to Leibniz through Oldenburg during 1676, both of which contain Eq. (9), with the explanation of its derivation appearing in the second letter. Although Newton seems to have been the first to deduce Eq. (9), he did not seem to recognize its significance, despite the fact that the notes in his "Waste Book" from 1664 show that he had independently rederived the curve of constant subtangent[48] (unaware of the work of Torricelli and Huygens) as the "instantaneous compound interest" curve.[54] Equation (9) was presented merely as a convenient means of obtaining a number given its logarithm.

In 1670 James Gregory[59] had obtained a close relative of Eq. (9) when he binomial-expanded the quantity

$$(1 + d/b)^{a/c} = 1 + (a/c)(d/b) + (a/c)(a/c - 1)(d/b)^2/2 + (a/c)(a/c - 1)(a/c - 2)(d/b)^3/6 + \cdots \quad (10)$$

which he applied to compound interest problems. Had he set $a = b$, $z = d/c$, and taken the limit of very large a he would have obtained Eq. (9) (plus one) without referring to a hyperbolic area or to a logarithm, a method used by Halley in 1695.[60]

F. Calculus formulation

Although the connection between the logarithmica and rate equations had in some sense been known since the Sumerians and was quantitatively formulated in terms of invariant subtangents by Torricelli and Huygens, the development of the calculus of Newton and Leibniz in the latter part of the 17th century provided the obvious context for its investigation. In 1676 Leibniz[61,62] read the letter of Descartes to de Beaune (then published) and found that the resulting equation

$$dY/dX = (Y - X)/b \quad (11)$$

could be transformed by the substitution $w = Y - X - b$ to the form

$$dw/dX = w/b. \quad (12)$$

This he then easily solved using the infinite series representation of Eq. (9) (with $x \rightarrow w - 1$ and $z \rightarrow X/b$). It is interesting to note that Leibniz had independently deduced Eq. (9) before he received Newton's letters through Oldenburg (Newton's letters were dated June 13 and October 24 of 1676, while Leibniz' notes of May 17, 1676 show his discovery of this equation[63]). However, Leibniz' statement of the solution to the de Beaune problem illustrates one of the reasons why knowledge of the exponential function in the 17th century is often underestimated. Leibniz stated that if w are numbers, X will be logarithms, rather than the inverse (that w will be logarithmicas if X are numbers). This tendency pervades the writings of Leibniz, Newton, and others of their time, and is often clumsy in physical applications in which the dependent and independent variables correspond, respectively, to a measured and a controlled quantity and are not arbitrary in their specification.

VI. THE 18th CENTURY: APPLICATIONS

The mathematical formulations of the logarithm and the logarithmica developed during the 17th century provided a means, not only to recognize exponential processes in nature, but also to measure their rate constants. The results of one measurement can then be used to predict the behavior of another. Although Huygens's attempts to describe an object's fall through a viscous medium did not agree with his measurements, there are a number of 18th century applications which were well described by the exponential law, and provide a useful context to follow the further development of this concept.

A. Newton's law of cooling

In 1701 Newton presented experimental results which indicated that the temperature of a heated object approaches that of its surroundings approximately exponentially in time. Newton published this paper[64] anonymously and in Latin and was perhaps purposefully enigmatic in his mathematically archaic exposition. The paper was mainly a description of the use of the cooling law to extend the range of temperature measurements above those accessible to standard thermometers by placing various metals on a heated iron and noting the cooling times at which the individual metals hardened. Newton stated in words that the heat loss should be proportional to the total heat, and from that deduced that the logarithm of the temperature should change uniformly with time, and suggested the use of a table of logarithms. Since time is the independent variable here, the temperature is given in terms of the inverse logarithm of the time, and it would have been more convenient to express the relationship in terms of the logarithmica. Despite his own considerable studies of this function 30 years earlier and the published results of Huygens and others which were available, Newton made no mention of the logarithmica or its properties.

B. Bouguer's law of absorption

An important example of an exponential process was discovered by Pierre Bouguer[65] in his studies of the attenuation of light by translucent materials in 1729. Bouguer illuminated adjacent surfaces by two identical light sources, one at a variable distance and the other with a translucent substance interposed, and used his eye as a null detector to establish equality of brightness. Correcting for the inverse square law, he found that the light diminishes in geometrical progression as the thickness of the translucent material increases in arithmetic progression. Familiar with the work of Huygens (whom he cited) and writing in French, he stated that the light diminishes as the "logarithmique" of the thickness of the absorber divided by a subtangent factor which is characteristic of the material. He included the infinite series representation for the logarithmique[65] [Eq. (9) above], but did not apply it to his actual numerical calculations, using instead Napier's logarithms to linearize the dependence. He included a set of examples showing how the subtangent could be deduced from experimental data for a given material and applied to other situations.

Bouguer's work was probably the last application of the exponential to be published before the introduction of our modern exponential notation, which was presented in Leonhard Euler's *Introductio*[29] in 1748. Euler had begun to use the letter e to represent the base of natural logarithms in 1727,[66] and in his *Introductio* he made frequent use of the equation

$$e^z = 1 + \frac{z}{1} + \frac{z^2}{1\times 2} + \frac{z^3}{1\times 2\times 3} + \frac{z^4}{1\times 2\times 3\times 4} + \cdots, \tag{13}$$

among other things, computing e to 23 decimal places. He also established the inverse operations $x = e^z$ and $z = \ln x$. Euler also devised the complex exponential, and identified the imaginary part with sinusoidal oscillations, making possible the mathematical formulation of damped harmonic motion. Euler's notation and the nomenclature "exponential" were quickly adopted and the terms "logarithmique" and "subtangent" were dropped from use in this context. This is apparent if we consider the work of Johannes Lambert,[67] who restudied and extended Bouguer's work in 1760. Although he followed Bouguer's formulation quite closely he made no mention of the logarithmique or the subtangent, using instead the Eulerian notation e^{ax}. It is interesting in this connection that many authors today refer to Bouguer's law as Lambert's law (or even as Beer's law after the 1851 work of August Beer) as though the law had been independently rediscovered by Lambert. Lambert merely cited the well-known work of Bouguer and applied it to specific situations and had no intention of claiming priority.

C. Prony's method and the thermal properties of fluids

In 1795 Riche de Prony[68] made a numerical analysis of data recorded by other investigators concerning the pressures, volumes, and temperatures of fluids. In some cases he noted apparent exponential relationships, and analyzed the data in terms of these functions. For example, he attempted to fit the measurements by Betancourt of the relation between temperature and vapor pressure to sums of from two to four exponentials. In doing this, Prony developed a method (still in use) in which the problem is reformulated in terms of two sets of linear equations and a single polynomial equation. The polynomial equation isolates the nonlinearities and its roots determine the fitting constants. The details of Prony's work have recently been reviewed by Bromage.[68] Applications involving sums of exponential functions became common in the nineteenth century, and are discussed in Sec. VII.

VII THE 19th CENTURY: DEPARTURES FROM A PURE EXPONENTIAL

As increasingly broad classes of phenomena were found to be well represented by the exponential law, small departures from this behavior were observed, and refinements in the mathematical models were developed in the 19th century which could accommodate these results. The use of linear rate equations is in some applications an approximation, and deviations from linearity must be considered. Even if the basic process is itself purely exponential in nature, the observed effect may involve a sum of exponential terms if several such processes are occurring simultaneously or sequentially. Although the characteristic shape common to all exponential functions simplifies analysis when seen alone, these generic similarities make it extremely difficult to disentangle the effects when several exponentials are seen together. In modern exponential mean-life determinations these effects are called "blending" and "cascade repopulation," and considerable effort has been devoted to avoiding and accounting for them during this decade[3] in fields such as atomic and molecular physics and biology. However, this problem had been formulated and the basic solutions had been obtained over 100 years ago.

A. Nonlinearities: the Verhulst curve

As was mentioned in Sec. IV, Pierre Verhulst attempted in 1838 to develop a mathematical model for the growth of a biological population. He began with a linear Malthusian rate equation, but added a quadratic retardation term to account for the struggle for survival, obtaining

$$dp/dt = ap - bp^2. \tag{14}$$

This has the solution

$$p(t) = [(1/p(0) - b/a)e^{-at} + b/a]^{-1} \tag{15}$$

which approaches an exponential law when b/a is small, but more generally is the s-shaped "logistic" curve. This curve describes not only biological situations, but also wide classes of nonlinear rate phenomena. The biological example provides an excellent illustration of the linear rate assumptions inherent in the exponential law, and their breakdown, since its validity sets in when the Fibonacci series assumes nearly the "golden ratio" and ends as the struggle for survival becomes the dominant dynamical process.

B. Dampled oscillations: the Kelvin LRC circuit

The damped oscillatory motion which is possible for an object possessing inertial mass and influenced by viscous drag and a Hooke's law linear restoring force is a rather obvious example of the exponential law, and was undoubtedly considered by many early observers. The electrical analog was studied by William Thompson[69] (Lord Kelvin) in 1853. Solving the differential equation

$$d^2q/dt + 2a\,dq/dt + b^2q = 0 \tag{16}$$

he obtained

$$q(t) = q(0)e^{-at}\cos[(b^2 - a^2)^{1/2}t + \phi]. \tag{17}$$

He pointed out the three now familiar cases: $a = b$, pure exponential decay; $a < b$, exponentially damped sinusoidal oscillations; and $a > b$, a sum of two exponentials.

C. Blending: the Becquerel phosphoroscope

In 1860 Edmond Becquerel[70] (the father of the discoverer of radioactivity) performed a series of measurements of the decay times of luminescent phosphors. For this purpose he had invented a device he called the "phosphoroscope," in which a phosphor is placed between two rotating disks with alternating opaque and light-transmitting sectors. Thus he was able to chop both the exciting light falling on the sample and the luminescent light it subsequently emitted, with an adjustable delay time between them. By measuring the luminescent intensity with a photocell as a function of delay time, he was able to obtain the desired decay curves. By determining the semilogarithmic slopes of these decay curves he was able to deduce the luminescent mean lives (he called them "coefficients d'extinction") for all but a few cases in which the semilogarithmic slope was not constant over the entire curve. He then assumed that what he was observing was a blend of two phosphors with two different mean lives, and, at least in one case, was able to extract both mean lives from the measured intensities I by adjusting the constants i_0, y_0, a, and b in the formula[71]

$$I = i_0e^{-at} + y_0e^{-bt}. \tag{18}$$

He commented that if the phosphor contains a large number of groups of rays of unequal persistences the calculation becomes too complicated to be compared with experiment (a problem which still exists today, despite advances in computational technology[3]). In some cases he found that the data were not describable by either exponential or multiexponential forms, but instead decayed in proportion to the reciprocal of the time since excitation. He showed that this situation would result if the decay rate were proportional to the square of the amount present, which suggests that a homogeneous collision driven mechanism dominated in these cases.

D. Cascading: the Esson equations

The exponential nature of certain chemical reactions had been reported already in 1850 by Ludwig Wilhelmy.[72] In studies of the inversion of sugar by acids he had used a polariscope to determine the reaction rate as a function of time, and found that in an excess of acid the rate is proportional to the amount of sugar present. (This is called "monomolecular change." If the reaction is limited by the amount present of more than one molecule, other equations pertain.) He presented the differential equation, its exponential solution, and determined the rate constants from logarithmic differences. Another set of experiments was carried out in 1865 by A. Vernon Harcourt and William Esson[73] which, although similar to those of Wilhelmy, contained some complications which exposed additional properties of the exponential decay process. Harcourt and Esson chose to study a chemical reaction which had easily controllable conditions and which could be abruptly stopped and its residual reactants accurately determined by chemical analysis. In exchange for these advantages, they accepted the disadvantages that the process was complicated by competing multistep reactions, with intermediate products which behaved like the original reactants except that their subsequent reaction proceeded with an additional rate constant. The experimental work was carried out primarily by Harcourt, who obtained a most perplexing group of decay curves possessing a variety of nonexponential humps and curvatures. The mathematical analysis of the linear rate equations (and nonlinear rate equations, for cases where other reactants were not in excess) was presented in a very thorough and complete appendix by Esson. Esson demonstrated that a reactant which has competing reaction channels decays by the same exponential law as one which has a single reaction mode, except that the exponential rate constant becomes the sum of the partial rates to the various modes. Thus all decay channels from the same parent decay with the same mean life. He considered the decay scheme shown in Fig. 4. Here a primary reactant $u(t)$ simultaneously undergoes two reactions with rates α and β. The total rate constant is then $\alpha + \beta$, and, for an initial amount $u(0) = a$,

$$u(t) = a\,e^{-(\alpha+\beta)t}. \tag{19}$$

He next noted that a sum of exponentials can be obtained in two distinct circumstances: when simultaneous independent processes cannot be experimentally distinguished (blending) as had been measured by Becquerel, and when a multistep sequential process causes both the formation and the decay of a reactant to proceed exponentially (cascading). He solved the coupled rate equations for the sequential chain of reactions affecting the intermediate reactant $v(t)$ in Fig. 4, which is formed with a rate β from $u(t)$ and undergoes a further reaction with rate γ, and obtained, assuming $v(0) = 0$, the Esson equation for a directly cascaded level,

$$v(t) = \{a\beta/[(\alpha + \beta) - \gamma]\}[e^{-\gamma t} - e^{-(\alpha+\beta)t}] \tag{20}$$

which exhibits the humped growing-in behavior which is a possible feature of a sequential process [it does not occur,[3] e.g., if $v(0)/u(0) > \beta/(\gamma - \alpha - \beta) > 0$] Harcourt and Esson's situation involved not only branching and cascading, but also blending, since their analysis of the residues could not distinguish between the primary and intermediate reactants u and v, and yielded only their sum

$$u(t) + v(t) = \{a/[(\alpha + \beta) - \gamma]\} \times [\beta\,e^{-\alpha t} + (\alpha - \gamma)\,e^{-(\alpha+\beta)t}]. \tag{21}$$

This implies either a monotonically decreasing ($\alpha \geq \gamma$) or a humped growing-in ($\alpha < \gamma$) decay. By using various temperatures and amounts of acid Harcourt and Esson

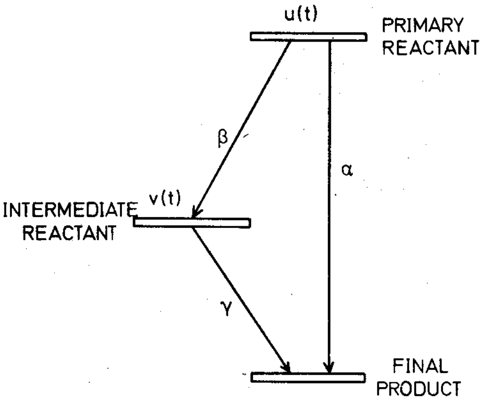

Fig. 4. Schematic representation of the chemical reactions studied by Harcourt and Esson in 1865. The chemical analysis of residues could not distinguish between the primary and intermediate reactants, $u(t)$ and $v(t)$, and only their sum was obtained, which involves all three rate constants α, β, and γ. Depending upon the relative values of the rate constants the decay curve could be single exponential, a monotonically decreasing sum of two exponentials, or a humped growing-in difference between two exponentials.

obtained single exponential, double exponential monotonic, and double exponential growing-in decay curves, and deduced values for the exponential rate constants from their measurements by a laborious numerical adjustment of the constants a, α, β, and γ. However, they stated that the number and exactness of the data were not sufficient to make the extracted rate constants more than approximate, which frustrated their desire to determine the relationship between rates and reactant concentrations.

E. Growing-in: the Walker ambiguity

Esson's mathematical analysis was reexamined in 1898 by James Walker.[74] Suppressing branching of $u(t)$ by setting $m = \alpha + \beta$ and $A = a\beta$ he rewrote Eq. (20) in the form

$$v(t) = A\,[e^{-\gamma t}/(m-\gamma) + e^{-mt}/(\gamma - m)] \qquad (22)$$

and recognized what we now call the growing-in ambiguity,[3] that is, Eq. (22) is completely symmetric in γ and m, and it is not possible to determine from $v(t)$ alone which process occurs *first*. He also suggested that if one of the processes is very fast it could very easily be overlooked, since its only influence would be for a very short time at the beginning of the reactions. Thus the process could appear to proceed with a single exponential rate constant corresponding to the slower of the two sequential processes.

VIII. CONCLUSION

The perceptions and applications of the exponential law which have been presented are by no means complete, and many other pre-20th-century examples could be cited. Further, the combined chronological and conceptual exposition which has been utilized here is more a convenient framework than a logical development, since exponential properties seem to have been rediscovered independently more often than they have been retrieved from previous works. However, the examples presented do clearly demonstrate that exponential change is a concept particularly well suited to human comprehension, requiring little mathematical sophistication and possessing a history of which radioactive decay comprises a rather small and unoriginal portion.

ACKNOWLEDGMENTS

This study was made possible through the access to the original books, periodicals, and other documents which was provided by the libraries of the Royal Academy of Sciences in Stockholm, of the University of Lund, and of the University of Uppsala. I am very grateful to Maurice Lambert of the Musée du Louvre for his generous help in the photographing of the cuneiform tablet, and to Dr. Gordon E. Bromage for making his work available prior to its publication.

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