Chapter 8: Potential Energy and The Conservation of Total Energy

Work and kinetic energy are energies of *motion*.

$$\Delta K = K_f - K_i = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = W_{net} = \int_{r_i}^{r_f} \vec{F} \cdot d\vec{r}$$

Potential energy is an energy that depends on *location*.

1-Dimension	3-Dimensions
$F_x = -\frac{dU(x)}{dx}$	$\vec{F}(\vec{r}) = -\nabla U(\vec{r}) = -\left[\frac{\partial U(\vec{r})}{\partial x}\hat{i} + \frac{\partial U(\vec{r})}{\partial y}\hat{j} + \frac{\partial U(\vec{r})}{\partial z}\hat{k}\right]$

Force is the derivative of Potential Energy:

$$F(x) = -dU(x)/dx$$

Thus, the force in the x-direction is the negative derivative of the potential energy! The same holds true for y- and z-directions.

$$F_x(x,y,z) = -\frac{\partial U(x,y,z)}{\partial x}$$

$$F_y(x,y,z) = -\frac{\partial U(x,y,z)}{\partial y}$$

$$F_z(x,y,z) = -\frac{\partial U(x,y,z)}{\partial z}$$

$$\vec{F} = \left(-\frac{\partial U(x,y,z)}{\partial x}\right)\hat{i} + \left(-\frac{\partial U(x,y,z)}{\partial y}\right)\hat{j} + \left(-\frac{\partial U(x,y,z)}{\partial z}\right)\hat{k}$$

Work done by Spring Force -- Summary

Spring force is a conservative force $\vec{F} = -k\vec{x}$

Work done *by* the spring force:

$$W_s = \int_{xi}^{xf} \vec{F} \cdot d\vec{x} = \int_{xi}^{xf} (-kx\,\hat{x}) \cdot d\vec{x} = -k \int_{xi}^{xf} x\,dx\,(\hat{x} \cdot \hat{x})$$

$$= -\left[\frac{1}{2}kx_f^2 - \frac{1}{2}kx_i^2\right] = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2$$

If $|x_f| > |x_i|$ (**further away from** equilibrium position); $W_s < 0$

If $|x_f| < |x_i|$ (**closer to** equilibrium position); $W_s > 0$

Let $x_i = 0$, $x_f = x$ then $W_s = -\frac{1}{2} k x^2$

Elastic Potential Energy

Spring force is a conservative force $\vec{F} = -k\vec{x}$

$$\Delta U \equiv U_f - U_i = -W = \frac{1}{2}kx_f^2 - \frac{1}{2}kx_i^2$$

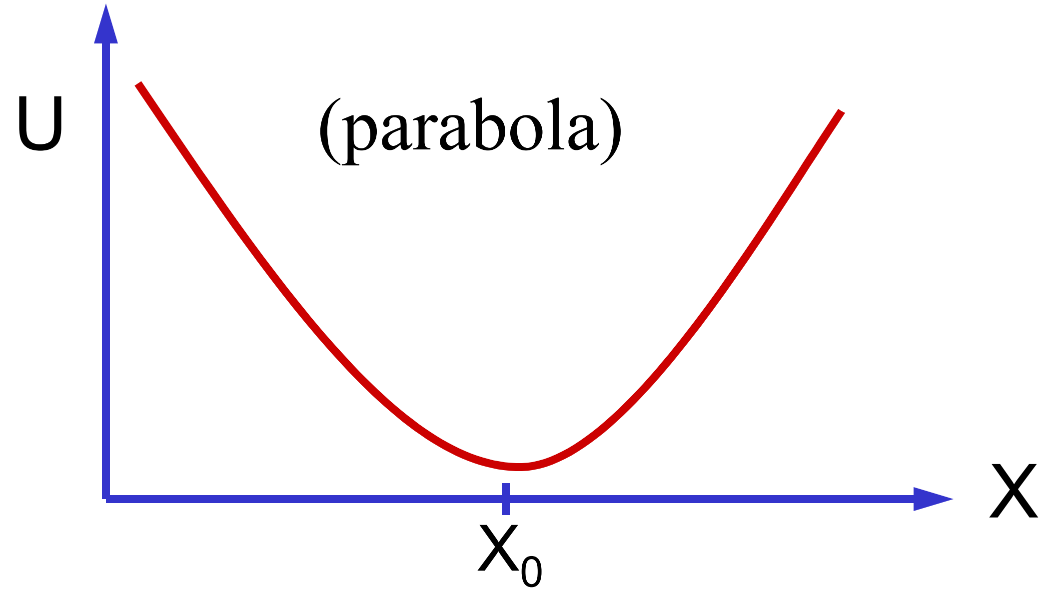
Choose the free end of the relaxed spring as the reference point: $U_i = 0$ at $x_i = 0$

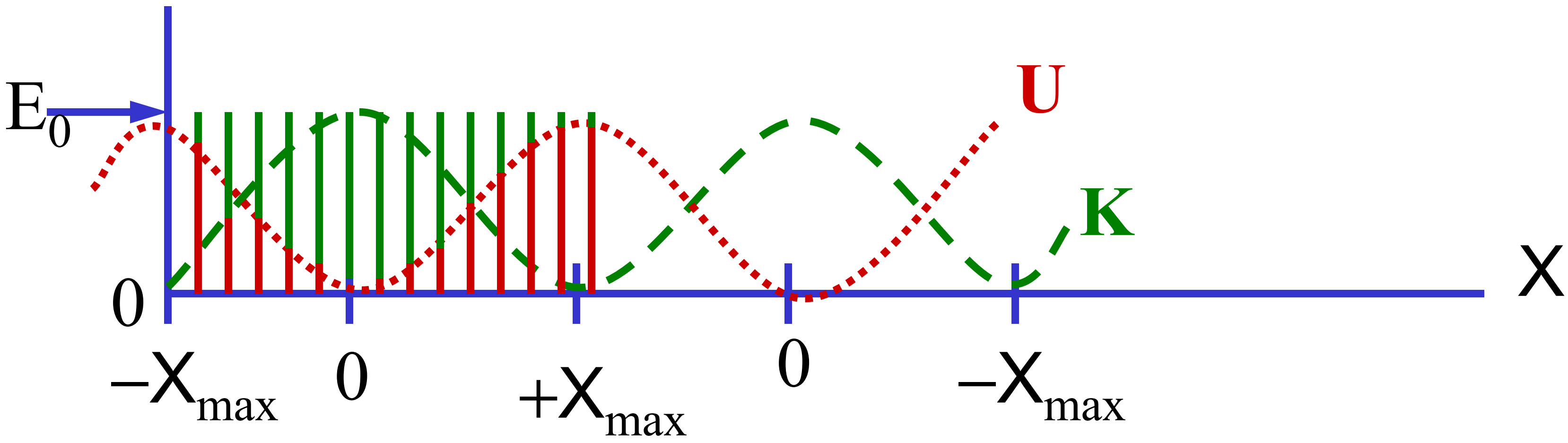
$$U(x) = -W(x) = \frac{1}{2}kx^2$$

The work went into potential energy, since the speeds are zero before and after.

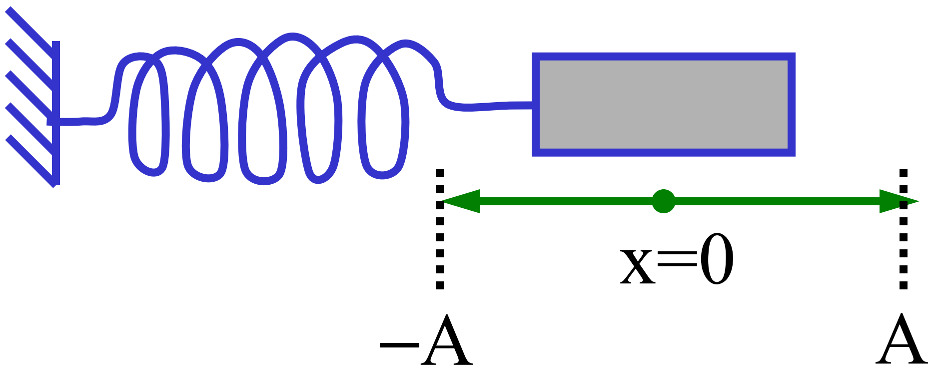
potential energy curve for springs

$$U(x) = -W(x) = \frac{1}{2}kx^2$$





Daily Quiz-1, February 08, 2006



$F = -kx$

$U = \int F\,dx = -\frac{1}{2}kx^2$

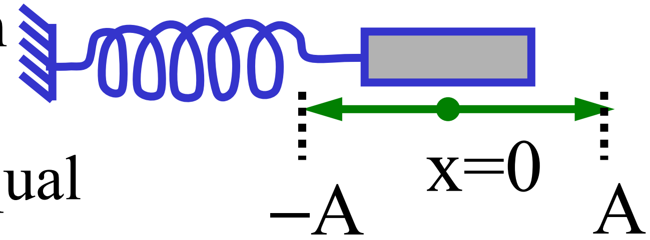
Horizontal spring with mass oscillating with maximum amplitude $|x_{max}| = A$. At which displacement(s) would the kinetic energy equal the potential energy?

1) $x = 0$ 2) $x = \pm A$ 3) $x = \frac{\pm A}{2}$ 4) $x = \frac{\pm A}{\sqrt{2}}$

5) none of the above

A Quiz

Horizontal spring with mass oscillating with maximum amplitude $|x_{max}| = A$. At which displacement(s) would the kinetic energy equal the potential energy?



1) $x = 0$ 2) $x = \pm A$ 3) $x = \frac{\pm A}{2}$ 4) $x = \frac{\pm A}{\sqrt{2}}$

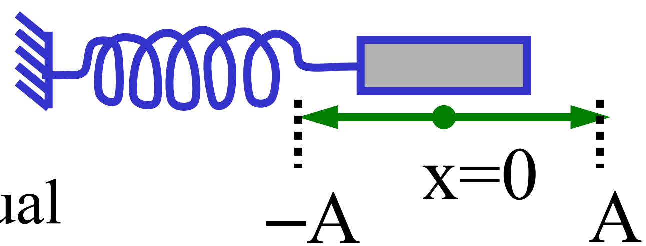
1. 1
2. 2
3. 3
4. 4
5. none of the above

A Quiz

$$E_{tot} = \frac{1}{2}kA^2 = K(x) + U(x) = 2(U(x)) = 2\left(\frac{1}{2}kx^2\right)$$

$$\Rightarrow \frac{1}{2}kA^2 = kx^2 \qquad \Rightarrow \quad x = \frac{\pm A}{\sqrt{2}}$$

Horizontal spring with mass oscillating with maximum amplitude $|x_{max}| = A$. At which displacement(s) would the kinetic energy equal the potential energy?

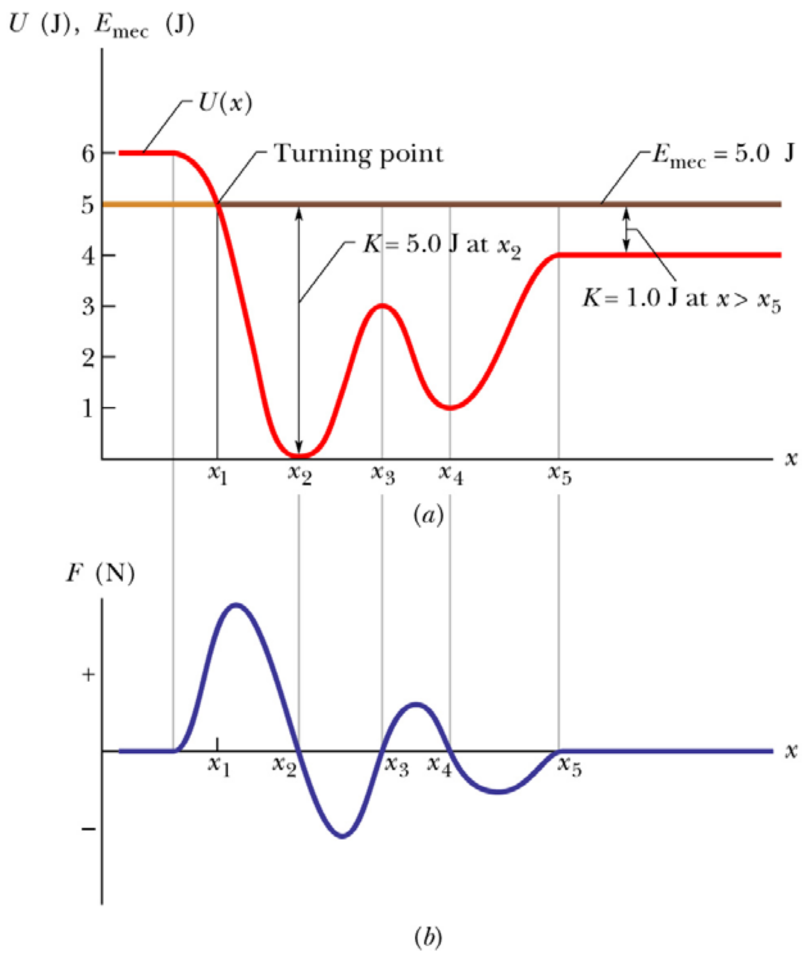


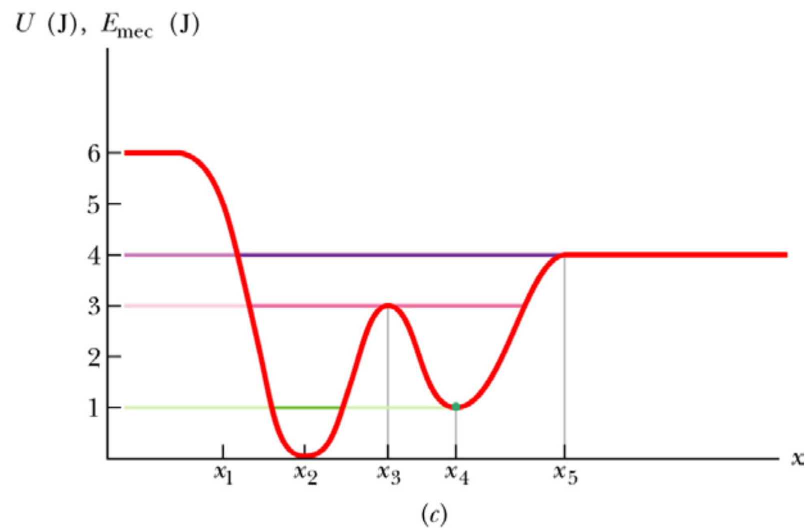
1) $x = 0$ 2) $x = \pm A$ 3) $x = \frac{\pm A}{2}$ 4) $x = \frac{\pm A}{\sqrt{2}}$

5) none of the above

consider a general potential energy curve-- U(x)

Thus, what "causes" a force is the variation of the potential energy function, i.e., the force is the negative derivative of the potential energy!

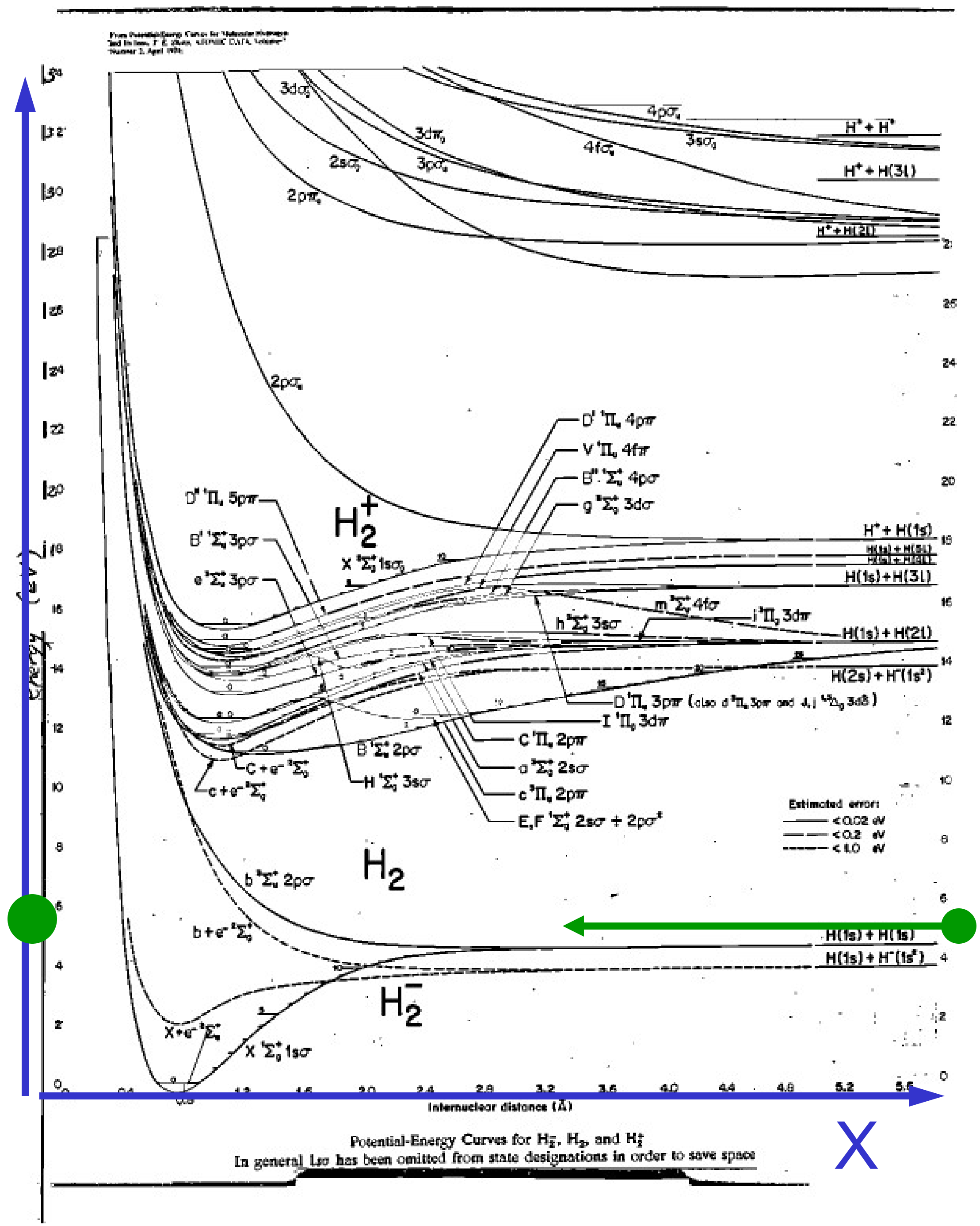




We know: $\Delta U(x) = -W = -F(x)\,\Delta x$

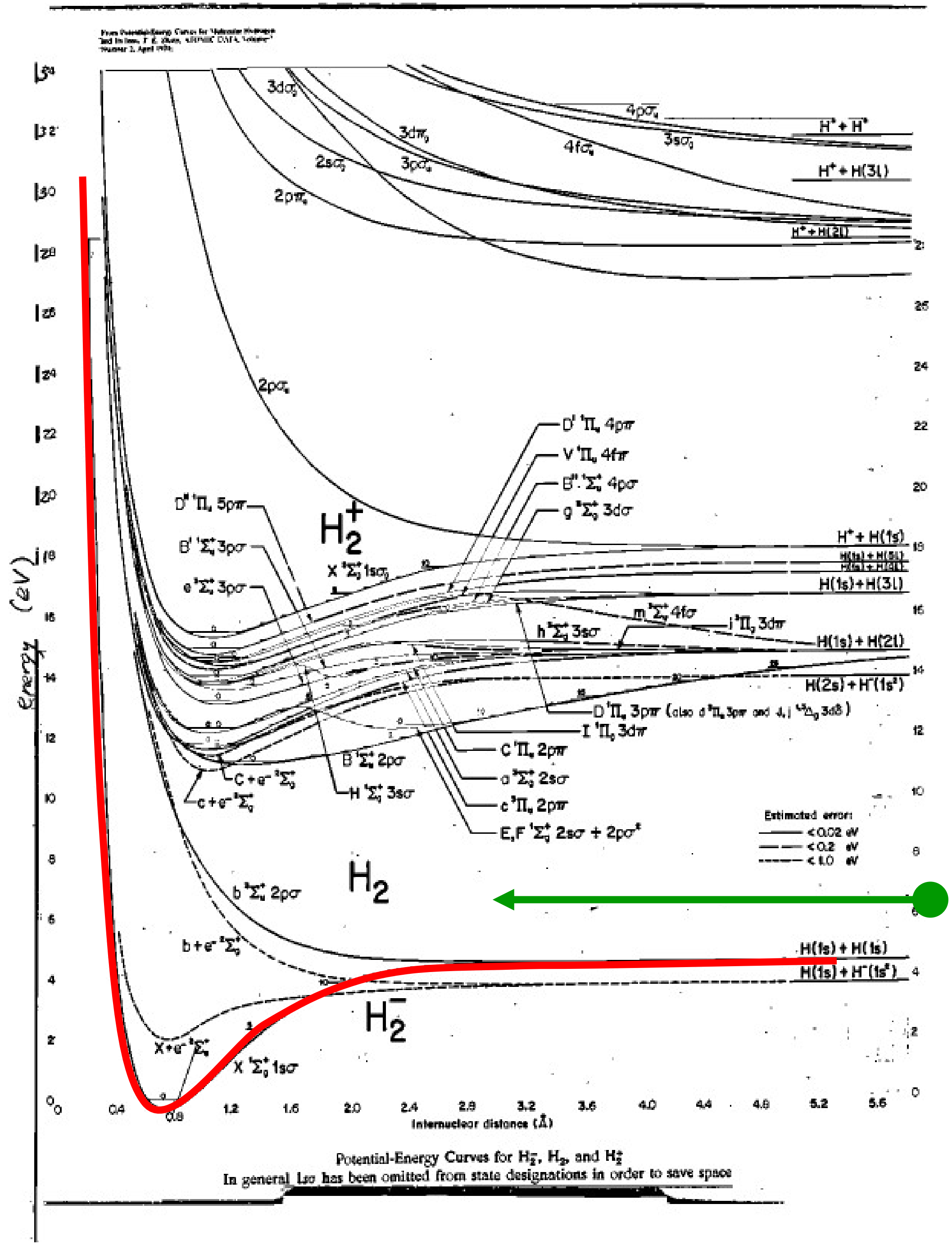
Therefore: $F(x) = -dU(x)/dx$

A hydrogen atom with kinetic energy of 4 eV is approaching another hydrogen atom in its ground state. The potential energy is shown to the right.



A hydrogen atom with kinetic energy of 4 eV is approaching another hydrogen atom in its ground state. The potential energy is shown to the right.

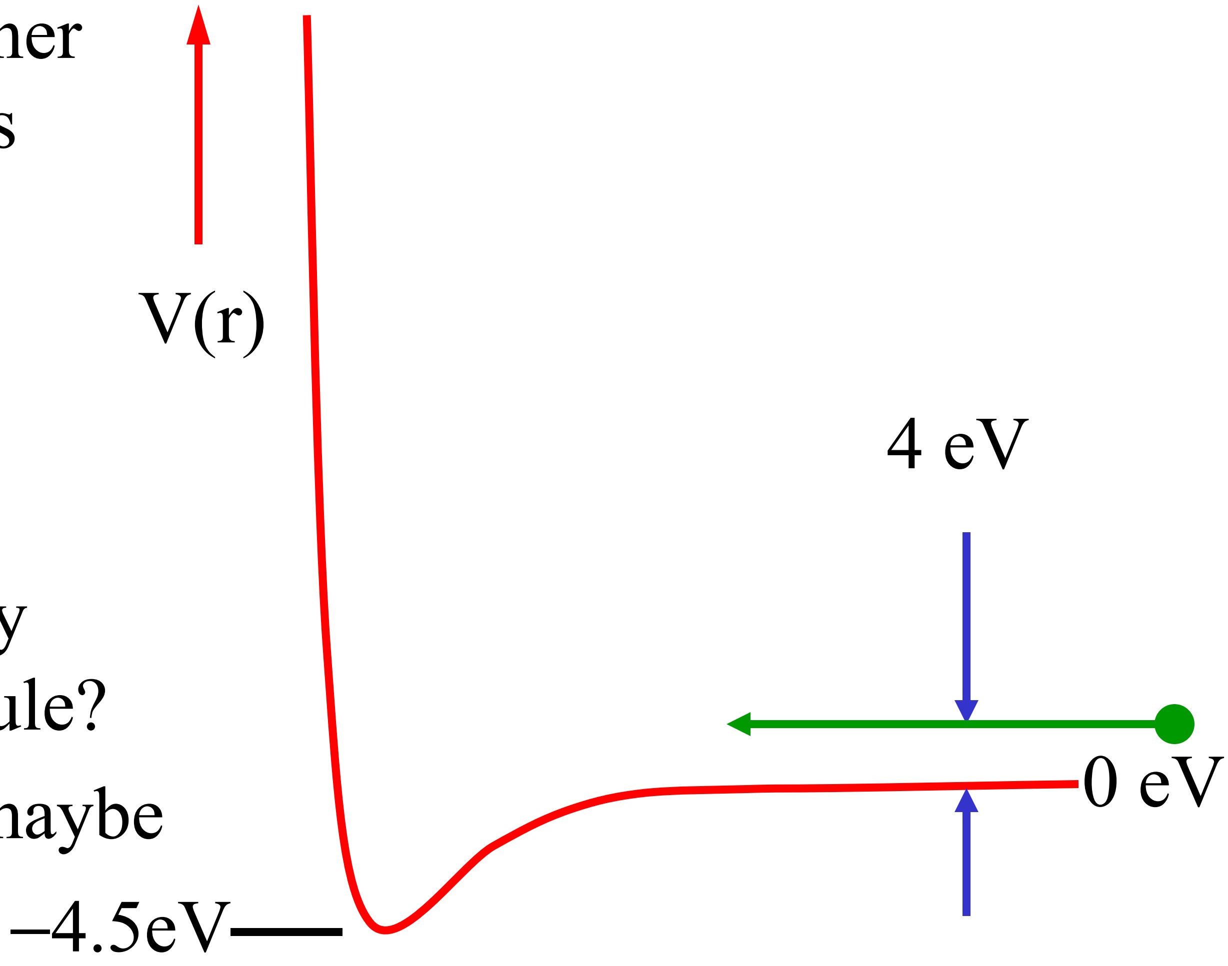
Will this H atom be captured and thereby become a H_2 molecule?



A hydrogen atom with kinetic energy of 4 eV is approaching another hydrogen atom in its ground state. The potential energy is shown to the right.

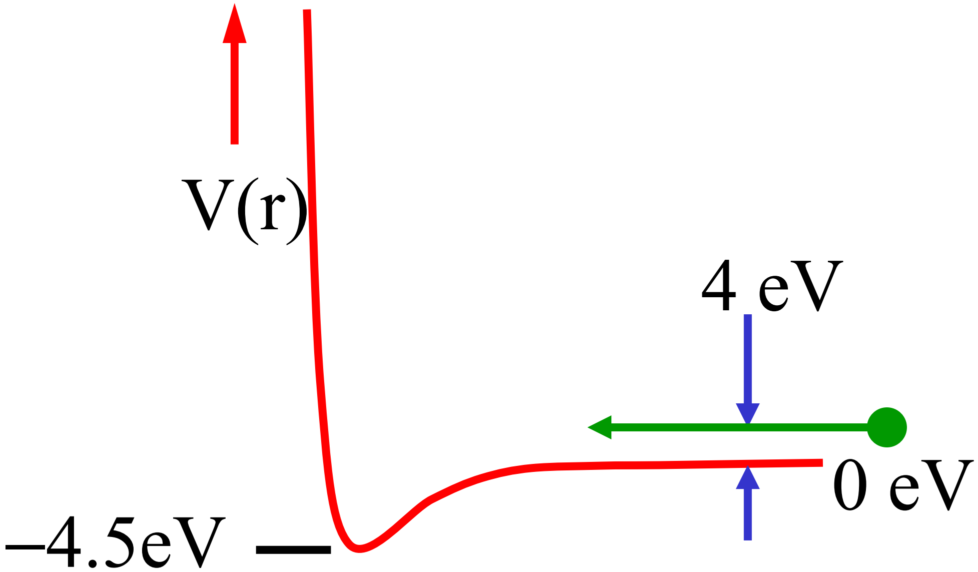
Will this H atom be captured and thereby become a H_2 molecule?

1) yes 2) no 3) maybe



Hydrogen Atom collisions

Will this H atom be captured and thereby become a H_2 molecule?



1. yes
2. no
3. maybe

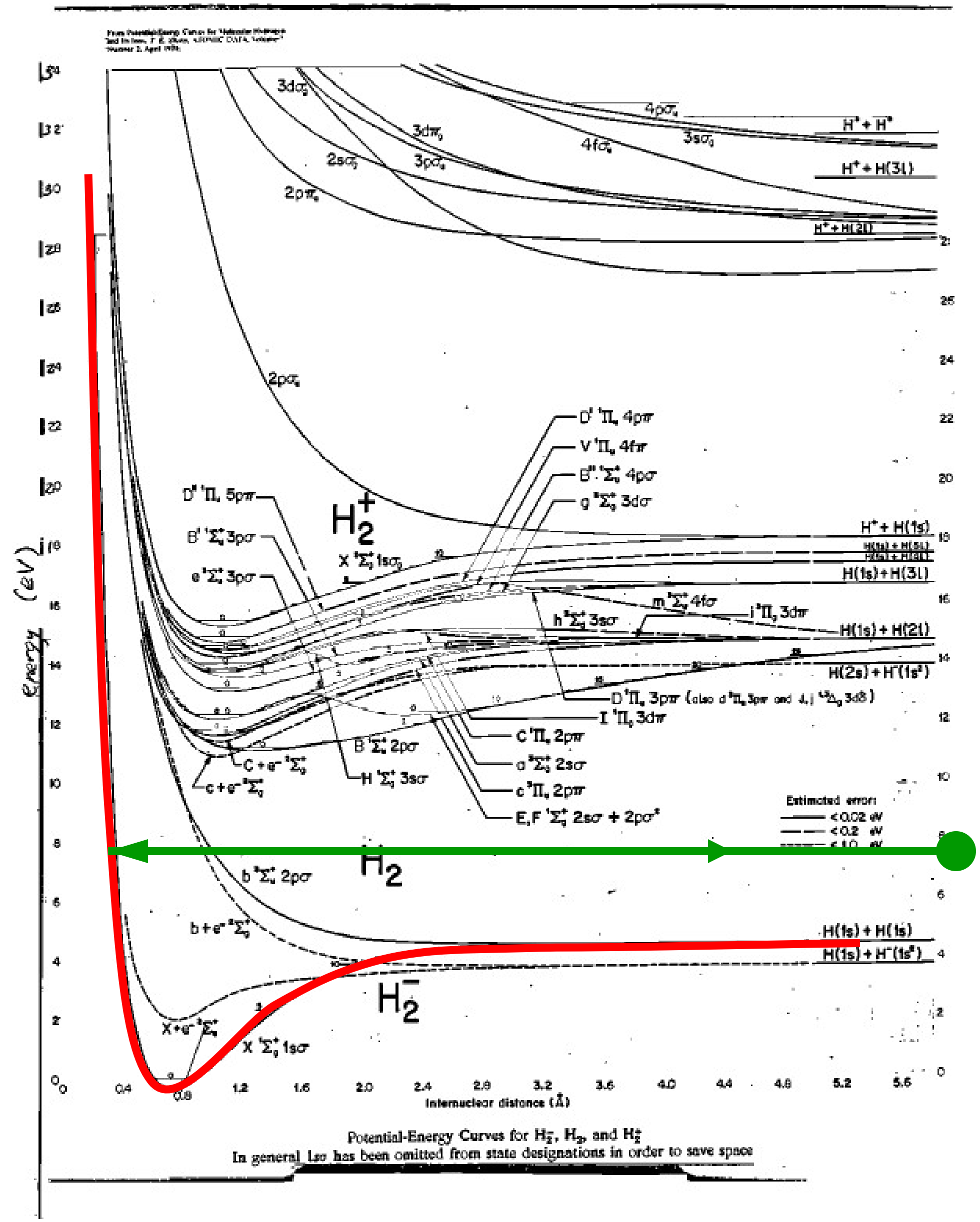
A hydrogen atom with kinetic energy of 4 eV is approaching another hydrogen atom in its ground state. The potential energy is shown to the right.

$$E_{initial} = K_{initial} + U_{initial}$$

4.0 eV

0 eV

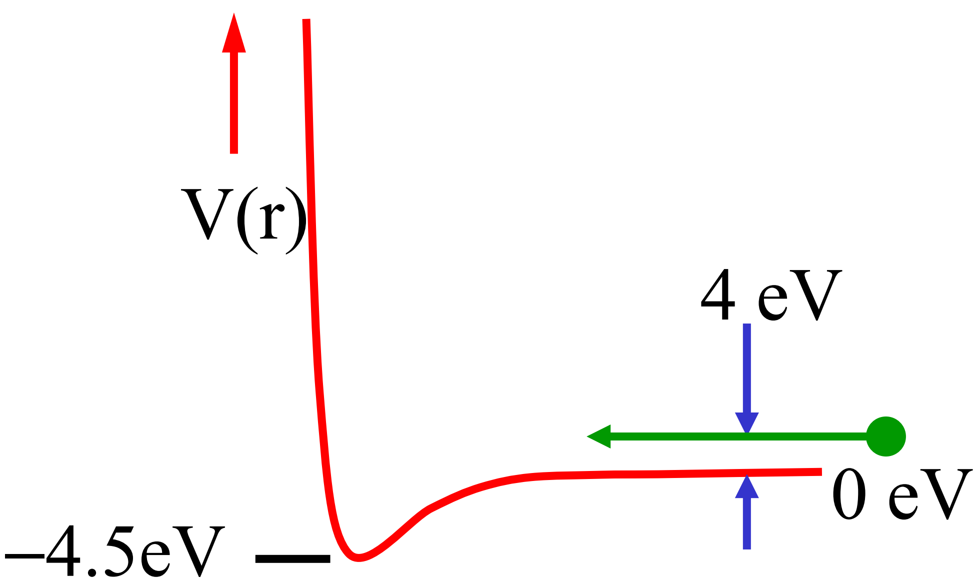
$= 4.0$ eV



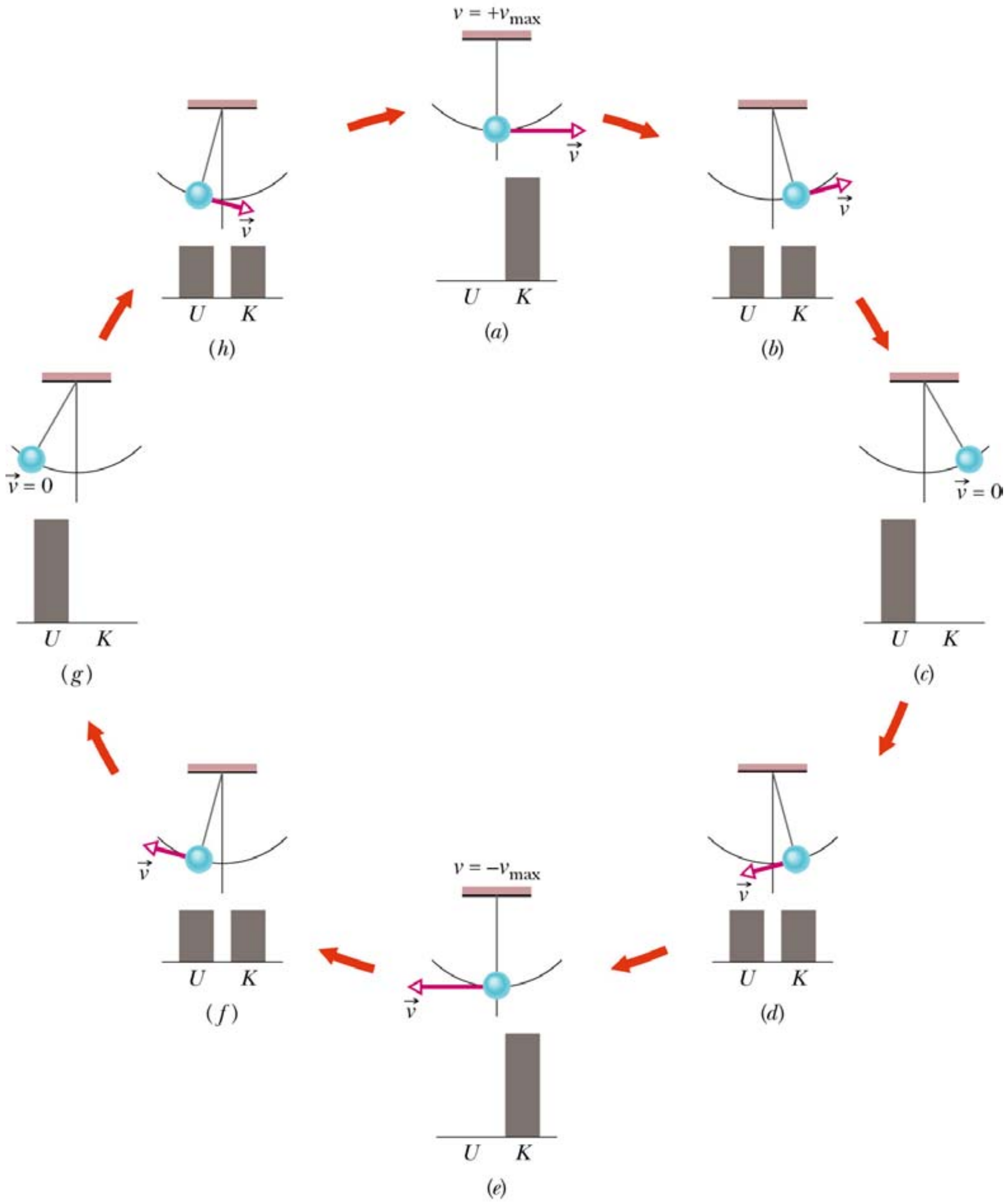
The H-atom hits the repulsive potential energy wall at about 0.04 nm and is reflected back to infinity. Another object is needed to absorb the excess kinetic energy. Note that the speed (kinetic energy) increases as the potential well becomes more negative, but the total energy is constant.

Will this H atom be captured and thereby become a H_2 molecule?

1) yes 2) no 3) maybe

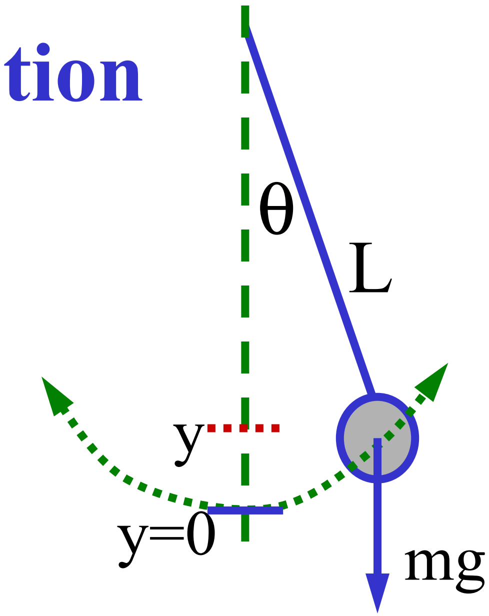


Pendulum and/or Spring



Mechanical energy is conserved ***if*** there are only conservative forces acting on the system.

analysis of pendulum motion

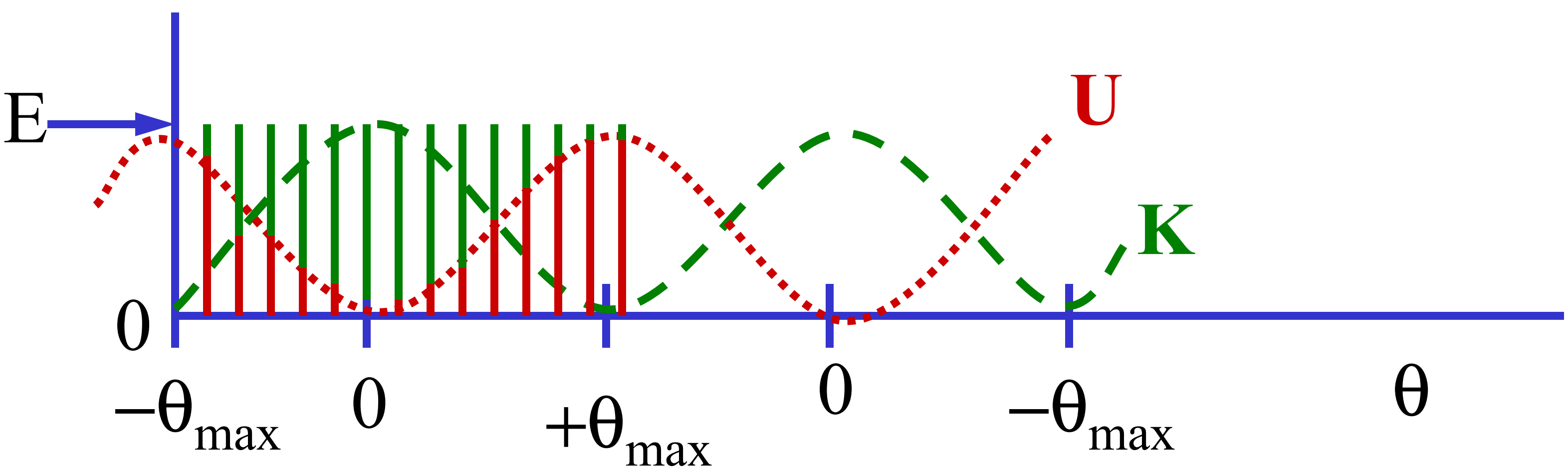


We can use mechanical energy to find the speed of the mass, m, as a function of angle:

Let the maximum height be y_{max} and y can be any height--

$$v = \sqrt{2g(y_{max} - y)} = \sqrt{2g\left(L(1-\cos\theta_{max}) - L(1-\cos\theta)\right)}$$

$$= \sqrt{2gL(\cos\theta - \cos\theta_{max})}$$



with conservation of energy, it is always important to specify the "system"

For an isolated system with only *conservative forces* (e.g., $F = mg$ and $F = -kx$) acting on the system:

$$E_{mec,1} = E_{mec,2} = E_{total} \quad \Rightarrow \quad K_1 + U_1 = K_2 + U_2 = E_{total}$$

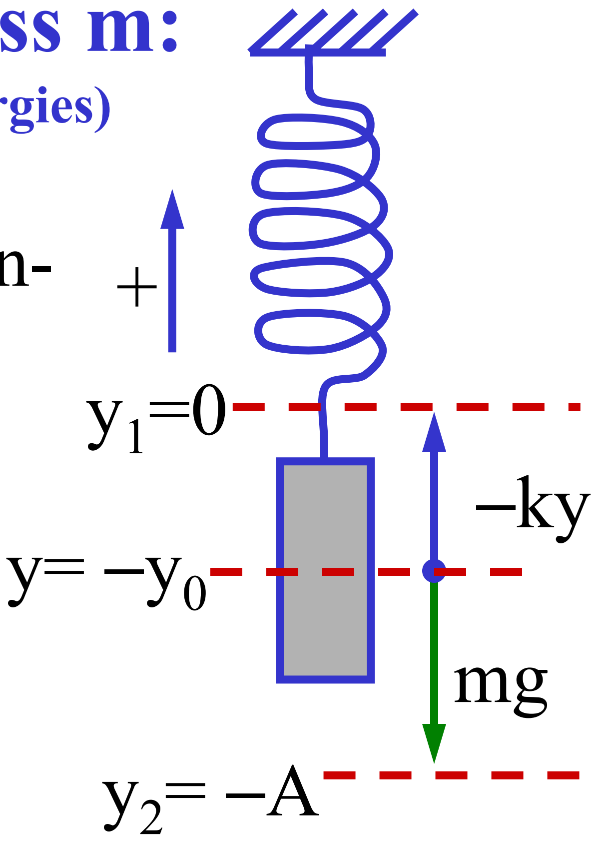
(initial) **(final)**

$$\tfrac{1}{2} m v_1^2 + m g x_1 + \tfrac{1}{2} k x_1^2 = \tfrac{1}{2} m v_2^2 + m g x_2 + \tfrac{1}{2} k x_2^2 = E_{total}$$

Initial mechanical energy   Final mechanical energy

Consider a vertical spring with mass m:

(we must consider both gravitational and elastic energies)



Assume that the mass m was held at the non-extended spring position (y_1=0) and then *slowly* brought down to the equilibrium position. Find this equilibrium position.

Equilibrium position ($F_g + F_s = 0$):

$F_g = -mg \quad F_s = -ky_0$

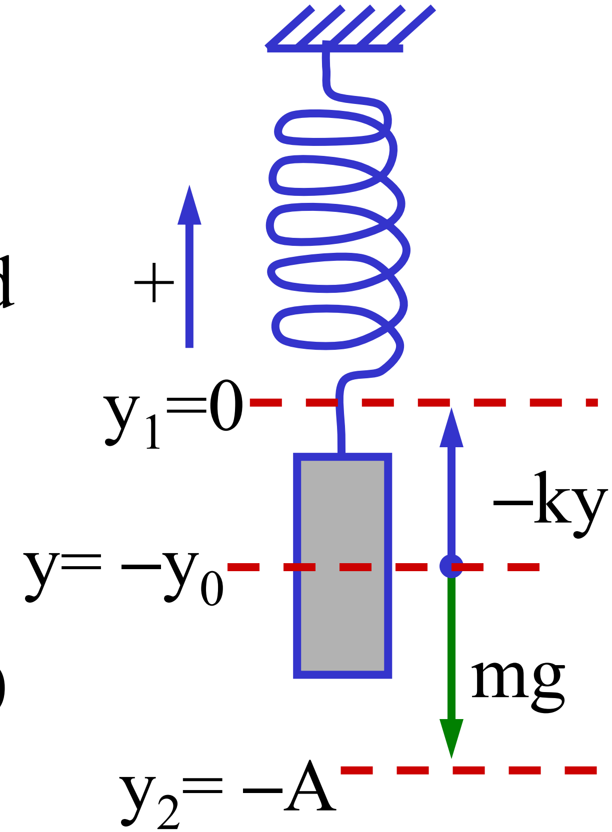
$\Rightarrow y_0 = -mg/k$

$K_e = 0, \quad U_{g,e} = mgy_0 = mg(-mg/k) = -(mg)^2/k,$

$U_{s,e} = ½\, ky_0^2 = ½\, k\,(-mg/k)^2 = ½\,(mg)^2/k$

vertical spring...

Assume that the mass m was held at the non-extended spring position ($y_1=0$) and then *let drop*. Find the lowest point the mass reaches.



Choose = 0

Initial position:

$K_1 = 0, \quad U_{g,1} = 0, \quad U_{s,1} = \frac{1}{2} ky^2 = 0$

0

$\Rightarrow E_{mech,1} = 0$

Final position:

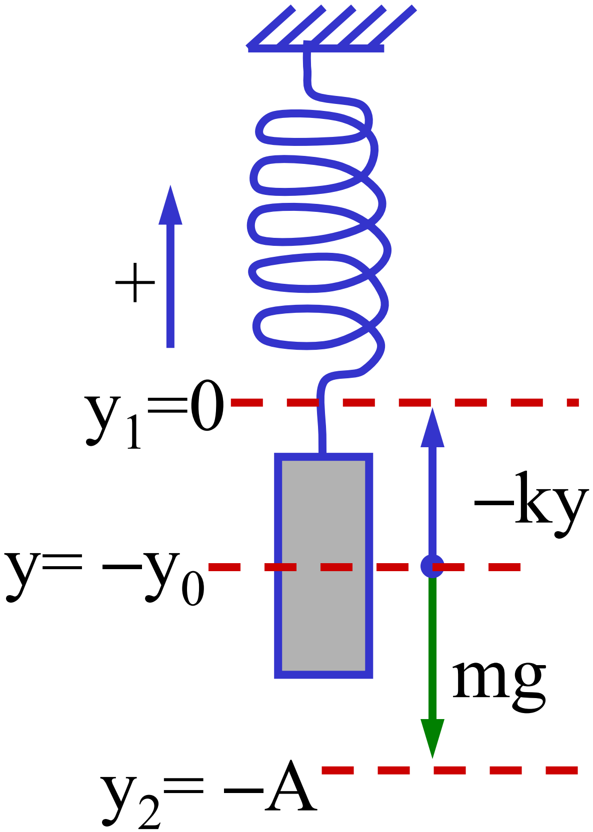
$K_2 = 0, \quad U_{g,2} = mgy_2 = mg(-A) = -mgA,$

$U_{s,2} = \frac{1}{2} ky_2^2 = \frac{1}{2} kA^2$

$$E_{mech,1} = E_{mech,2} = 0 = U_{g,2} + U_{s,2} = -mgA + \frac{1}{2} kA^2$$

vertical spring...

Assume that the mass m was held at the non-extended spring position ($y_1=0$) and then *let drop*. Find the lowest point the mass reaches.



Final position:

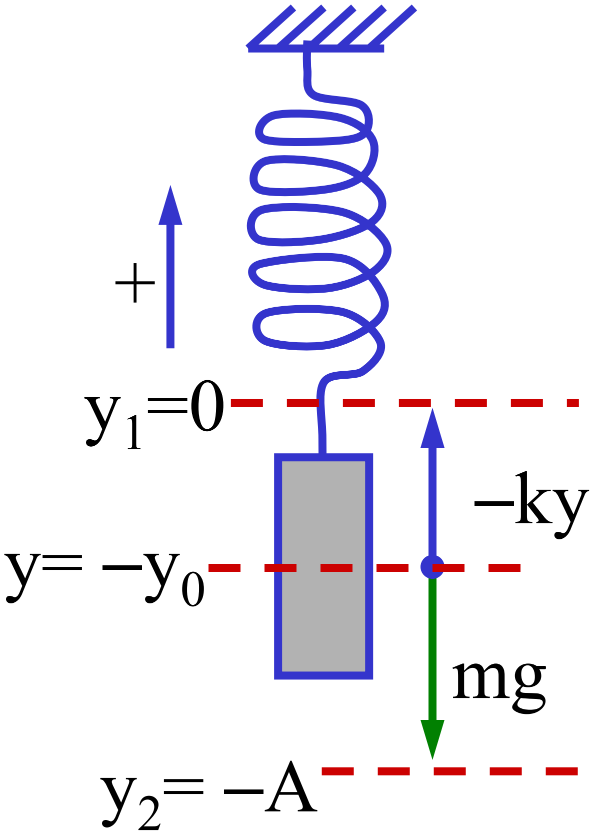
$$E_{mech,1} = E_{mech,2} = 0 = U_{g,2} + U_{s,2} = -mgA + \tfrac{1}{2} kA^2$$

$$\Rightarrow \quad A = 2mg/k \qquad (= 2y_0)$$

vertical spring...

Assume that the mass m was held at the non-extended spring position (y_1=0) and then *let drop*. Find the maximum speed of the mass.

(Maximum speed occurs when the potential energy is minimum.)



Arbitrary position:

$$E_{mech} = 0 = K + U_g + U_s = \tfrac{1}{2} mv^2 + mgy + \tfrac{1}{2} ky^2$$

$$dU/dy = d(mgy + \tfrac{1}{2} ky^2)/dy = 0$$

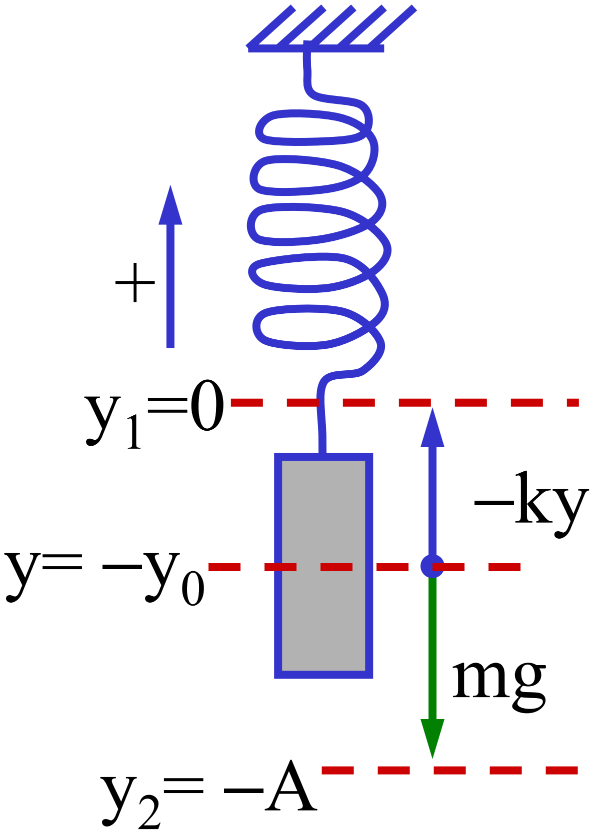
=> $mg + ky = 0$ => $y = -mg/k = y_0$

=> maximum speed occurs at the equilibrium point

vertical spring...

Assume that the mass m was held at the non-extended spring position ($y_1=0$) and then *let drop*. Find the maximum speed of the mass.

(Maximum speed occurs when the potential energy is minimum.)



Maximum speed position:

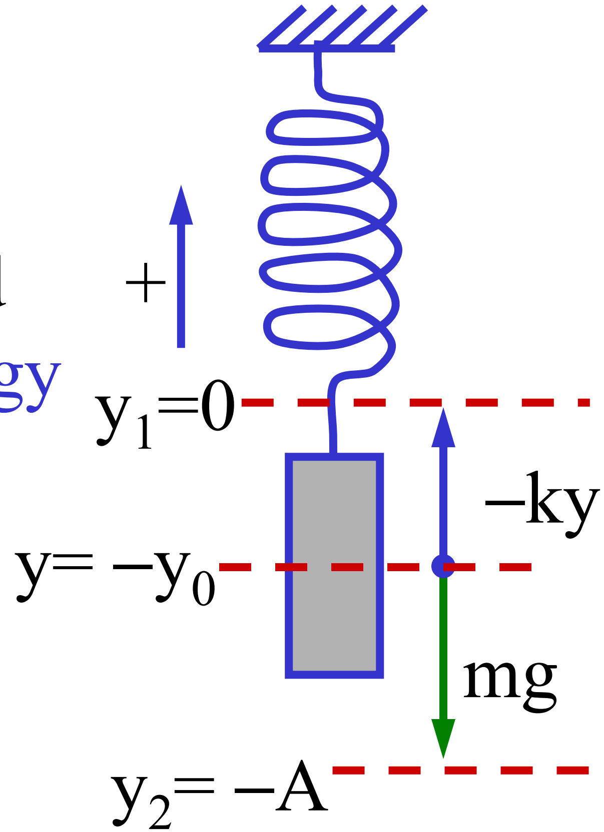
$$0 = K_{max} + U_g + U_s = \tfrac{1}{2} mv^2 + mg(-mg/k) + \tfrac{1}{2} k(-mg/k)^2$$

$$0 = K_{max} + U_g + U_s = \tfrac{1}{2} mv^2 - (mg)^2/k + \tfrac{1}{2}(mg)^2/k$$

$$= \tfrac{1}{2} mv^2 - \tfrac{1}{2} (mg)^2/k \quad \Rightarrow \quad v^2 = mg^2/k$$

vertical spring...

Assume that the mass m was held at the non-extended spring position (y_1=0) and then *let drop*. Find the mechanical energy of the mass at the equilibrium point.



Kinetic energy at $y = -y_0$:

$$K_{y0} = \tfrac{1}{2}\,(mg)^2/k$$

Potential energy at $y = -\,y_0$:

$$(U_g+U_s)|_{y=-y0} = mg(-y_0) + \tfrac{1}{2}\,k(-y_0)^2 = -mgy_0 + \tfrac{1}{2}k(y_0)^2$$

$$= -mg(mg/k) + \tfrac{1}{2}k(mg/k)^2 = -\tfrac{1}{2}(mg)^2/k$$

Mechanical energy at $y = -\,y_0$:

$$E_{mech} = K_{y0} + (U_g+U_s)|_{y=-y0} = \tfrac{1}{2}\,(mg)^2/k - \tfrac{1}{2}(mg)^2/k = 0$$

Work done by external force

When no friction acts within the system, the net work done by the external force equals the change in mechanical energy

$$W_{net} = \Delta E_{mec} = \Delta K + \Delta U$$

Friction is a **non-**conservative force that opposes motion

Work done by friction is: $W_{friction} = -f_k d$

When a kinetic friction force acts within the system, then the thermal energy of the system changes:

$$\Delta E_{th} = f_k d$$

Therefore $W = \Delta E_{mec} + \Delta E_{th}$

Work done by external forces

When there are **non-conservative** forces (like friction) acting on the system, the net work done by them equals the change in mechanical energy

$$W_{net} = \Delta E_{mec} = \Delta K + \Delta U$$

Conservation of Total Energy

The total energy E of a system can change only by an amount of energy that is transferred to or from the system.

$$W = \Delta E = \Delta E_{mec} + \Delta E_{th} + \Delta E_{int}$$

If there is no internal energy change, but friction acts within the system: $W = \Delta E_{mec} + \Delta E_{th}$

If there are only conservative forces acting within the system: $W = \Delta E_{mec}$

Isolated Systems

For an isolated system (W = 0), the total energy E of the system cannot change

$$\Delta E_{mec} + \Delta E_{th} + \Delta E_{int} = 0$$

For an isolated system with only conservative forces, ΔE_{th} and ΔE_{int} are both zero.

Therefore: $\Delta E_{mec} = 0$

Law of Conservation of Total Energy

$$E_i = K_i + U_i + E_{thermal_i} + E_{internal_i}$$

$$=$$

$$E_f = K_f + U_f + E_{thermal_f} + E_{internal_f}$$

Rearrange terms.

$$(E_f - E_i) =$$
$$(K_f - K_i) + (U_f - U_i) + (E_{thermal_f} - E_{thermal_i}) + (E_{internal_f} - E_{internal_i}) = 0$$

$$\Delta E = (\Delta K + \Delta U) + \Delta E_{thermal} + \Delta E_{internal}$$
$$= \Delta E_{mechanical} + \Delta E_{thermal} + \Delta E_{internal} = 0$$

Law of Conservation of Energy

Count up the initial energy in ***all*** of its forms.

$$E_i = K_i + U_i + E_{thermal_i} + E_{internal_i}$$

Count up the final energy in ***all*** of its forms.

$$E_f = K_f + U_f + E_{thermal_f} + E_{internal_f}$$

These two ***must*** be equal.

$$E_i = K_i + U_i + E_{thermal_i} + E_{internal_i}$$

$$= E_f = K_f + U_f + E_{thermal_f} + E_{internal_f}$$

Sample Problem 8-6

A wooden crate of $m = 14kg$ is pushed along a horizontal floor with a constant force of $|F| = 40N$ for a total distance of $d = 0.5m$, during which the crate's speed decreased from $v_o = 0.60$ m/s to $v = 0.20$m/s.

A) Find the work done by F.

$$W = Fd\cos\phi = (40N)(0.50m)\cos 0^o = 20J$$

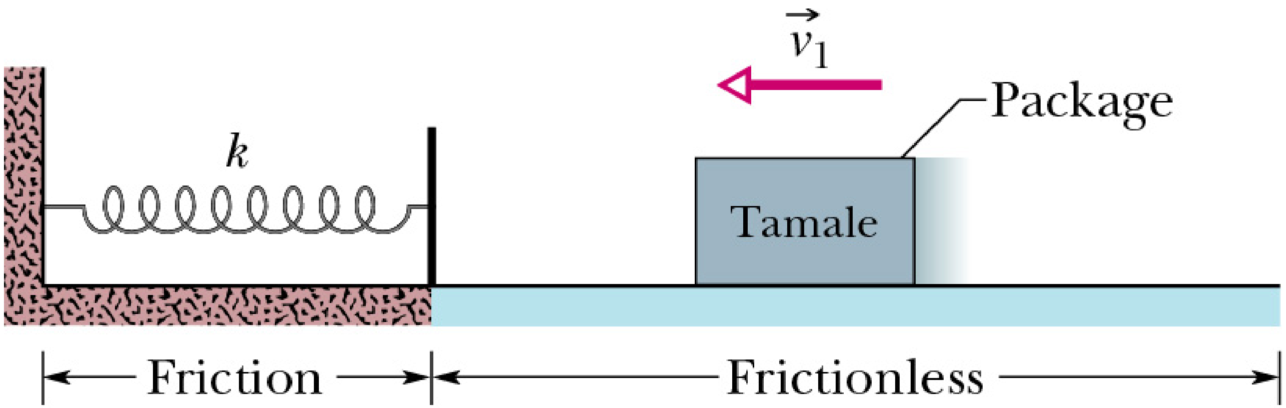
B) Find the increase in thermal energy.

$$W = \Delta E_{mec} + \Delta E_{thermal} = 20J = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_o^{\,2} + \Delta E_{thermal}$$

$$\Rightarrow \Delta E_{thermal} = W - \Delta E_{mec} = 20J - (\tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_o^{\,2}) = 22.2J$$

Sample Problem 8-7

In the figure, a 2.0kg package slides along a floor with speed v_1 = 4.0m/s. It then runs into and compresses a spring, until the package momentarily stops. Its path to the initially relaxed spring is frictionless, but as it compresses the spring, a kinetic friction force from the floor, of magnitude 15 N , act on it. The spring constant is 10,000 N/m. By what distance d is the spring compressed when the package stops?



The change in mechanical energy must equal the energy converted to thermal energy.

$$\Delta E_{mechanical} + \Delta E_{thermal} = 0$$
$$\Rightarrow (E_{mec,2} - E_{mec,1}) + \Delta E_{thermal} = 0$$

Initial mechanical energy,

$$E_{mec,1} = K_1 + U_1 = \tfrac{1}{2}mv_1^2 + U_1 = \tfrac{1}{2}mv_1^2 + 0 = \tfrac{1}{2}mv_1^2$$

Final mechanical energy,

$$E_{mec,2} = K_2 + U_2 = \tfrac{1}{2}mv_2^2 + U_2 = 0 + \tfrac{1}{2}kd_2^2 = \tfrac{1}{2}kd^2$$

The change in mechanical energy must equal the energy converted to thermal energy.

$$\Delta E_{thermal} = f_k d$$

$$\left(E_{mec,2} - E_{mec,1}\right) + \Delta E_{thermal} = \left(\tfrac{1}{2}kd^2 - \tfrac{1}{2}mv_1^2\right) + f_k d = 0$$

Thus, a quadratic equation in d with:

m = 2.0 kg, v_1 = 4.0 m/s, f_k = 15 N, and k = 10,000 N/m

$$\tfrac{1}{2}kd^2 + f_k d - \tfrac{1}{2}mv_1^2 = 0$$

$$\Rightarrow d = 0.055\text{m}$$

Potential Energy Curve

We know: $\Delta U(x) = -W = -F(x)\,\Delta x$

Therefore: $F(x) = -dU(x)/dx$

Now integrate along the displacement:

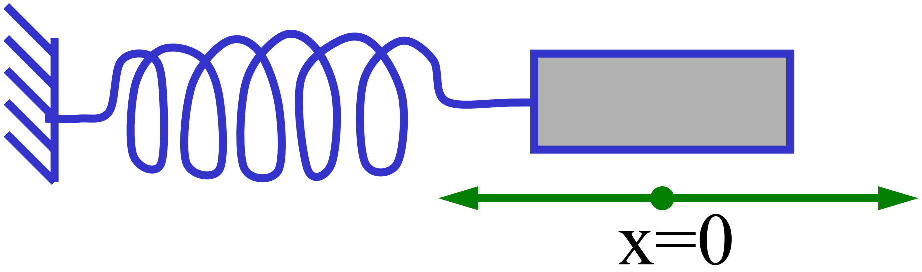
$$W = \int \vec{F} \cdot d\vec{x} = -\int \frac{dU}{dx}\,dx$$

$$\int \vec{F} \cdot d\vec{x} = K_f - K_i = -\int \frac{dU}{dx}\,dx = -(U_f - U_i) = U_i - U_f$$

$$\int \vec{F} \cdot d\vec{x} = -\int \frac{dU}{dx}\,dx \quad = \quad K_f - K_i = U_i - U_f$$

Rearrange terms: $K_f + U_f = K_i + U_i$

Horizontal Spring



Isolated system with only conservative forces acting on it.

(e.g., $\vec{F} = -k\vec{x}$)

$$E_{mech,1} = E_{mech,2} = E_{total}$$

$$K_1 + U_1 = K_2 + U_2 = E_{total}$$

$$\tfrac{1}{2}\, mv_1^2 + \tfrac{1}{2}\, kx_1^2 = \tfrac{1}{2}\, mv_2^2 + \tfrac{1}{2}\, kx_2^2 = E_{total}$$