Chapter 4

Motion in Two and Three Dimensions

In this chapter we will continue to study the motion of objects without the restriction we put in chapter 2 to move along a straight line. Instead we will consider motion in a plane (two dimensional motion) and motion in space (three dimensional motion)
The following vectors will be defined for two- and three- dimensional motion:

Displacement
Average and instantaneous velocity
Average and instantaneous acceleration

We will consider in detail projectile motion and uniform circular motion as examples of motion in two dimensions

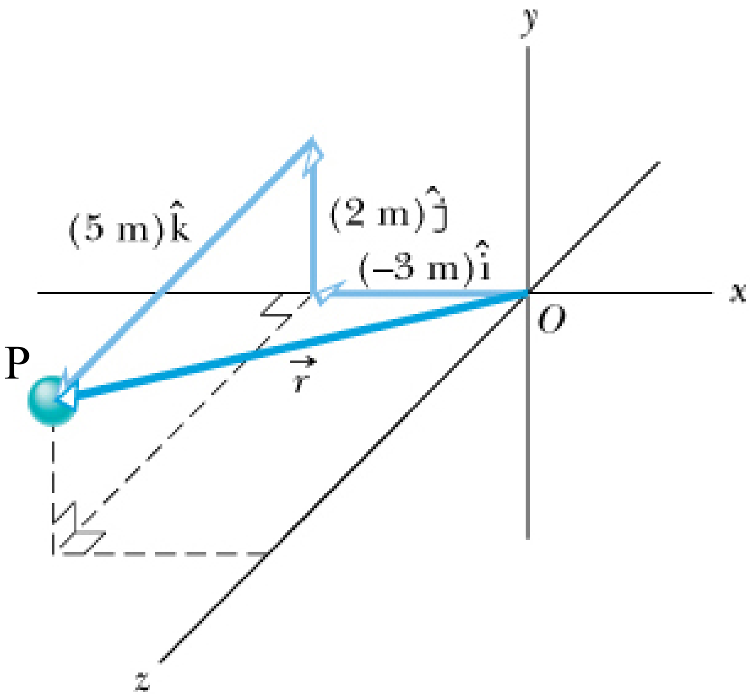
Finally we will consider relative motion, i.e. the transformation of velocities between two reference systems which move with respect to each other with constant velocity

Position Vector

The position vector $\vec{r}$ of a particle is defined as a vector whose tail is at a reference point (usually the origin O) and its tip is at the particle at point P.

Example: The position vector in the figure is:

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$



$$\vec{r} = \left(-3\hat{i} + 2\hat{j} + 5\hat{k}\right) m$$

Displacement Vector

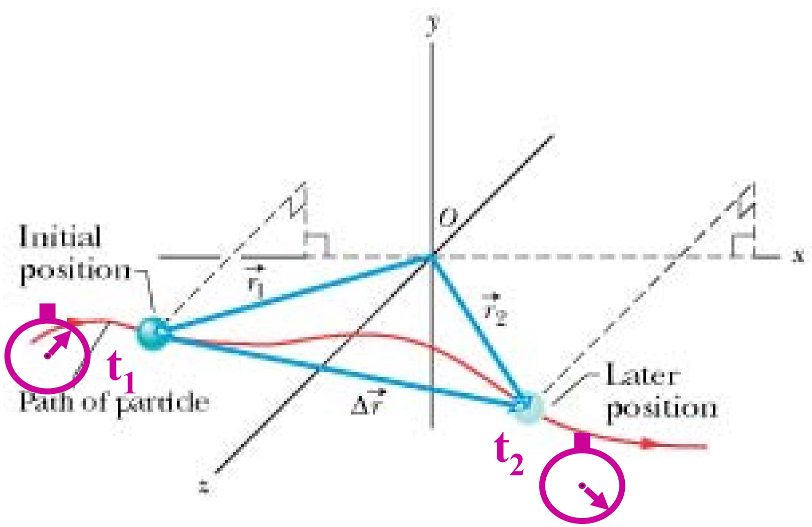
For a particle that changes postion vector from $\vec{r}_1$ to $\vec{r}_2$ we define the displacement vector $\Delta\vec{r}$ as follows: $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$

The position vectors $\vec{r}_1$ and $\vec{r}_2$ are written in terms of components as:

$\vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ $\vec{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

The displacement $\Delta\vec{r}$ can then be written as:

$$\Delta\vec{r} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k} = \Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}$$



$$\Delta x = x_2 - x_1$$

$$\Delta y = y_2 - y_1$$

$$\Delta z = z_2 - z_1$$

Average and Instantaneous Velocity

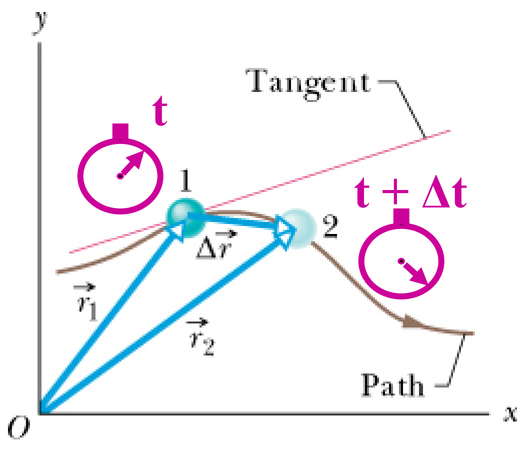
Following the same approach as in chapter 2 we define the average velocity as:

$$\text{average velocity} = \frac{\text{displacement}}{\text{time interval}}$$

$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}}{\Delta t} = \frac{\Delta x\hat{i}}{\Delta t} + \frac{\Delta y\hat{j}}{\Delta t} + \frac{\Delta z\hat{k}}{\Delta t}$$

We define as the instantaneous velocity (or more simply the velocity) as the limit:

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$



If we allow the time interval Δt to shrink to zero, the following things happen:

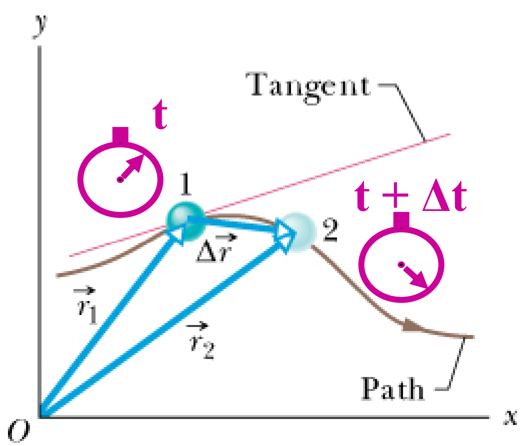
1. Vector $\vec{r}_2$ moves towards vector $\vec{r}_2$ and $\Delta\vec{r} \to 0$

2. The direction of the ratio $\dfrac{\Delta\vec{r}}{\Delta t}$ (and thus $\vec{v}_{avg}$)approaches the direction of the tangent to the path at position 1

3. $\vec{v}_{avg} \to \vec{v}$

$$\vec{v} = \frac{d}{dt}\left(x\hat{i} + y\hat{j} + z\hat{k}\right) = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

The three velocity components are given by the equations:



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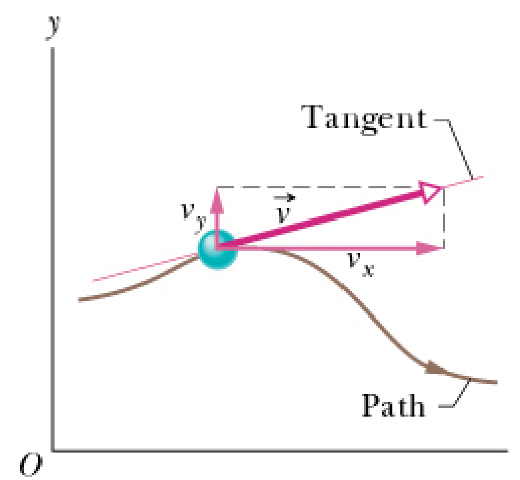
$$\vec{v} = \frac{d\vec{r}}{dt}$$

$$v_x = \frac{dx}{dt}$$

$$v_y = \frac{dy}{dt}$$

$$v_z = \frac{dz}{dt}$$



Average and Instantaneous Acceleration

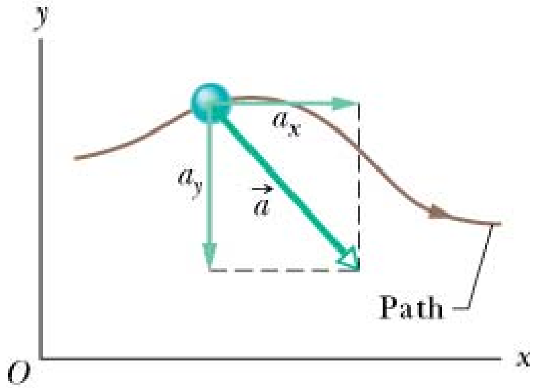
The average acceleration is defined as:

$$\text{average acceleration} = \frac{\text{change in velocity}}{\text{time interval}} \qquad \vec{a}_{avg} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t} = \frac{\Delta \vec{v}}{\Delta t}$$

We define as the instantaneous acceleration as the limit:

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(v_x \hat{i} + v_y \hat{j} + v_z \hat{k}\right) = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$$

Note: Unlike velocity, the acceleration vector does not have any specific relationship with the path.



The three acceleration components are given by the equations:

$$a_x = \frac{dv_x}{dt} \qquad a_y = \frac{dv_y}{dt} \qquad a_z = \frac{dv_z}{dt} \qquad \vec{a} = \frac{d\vec{v}}{dt}$$

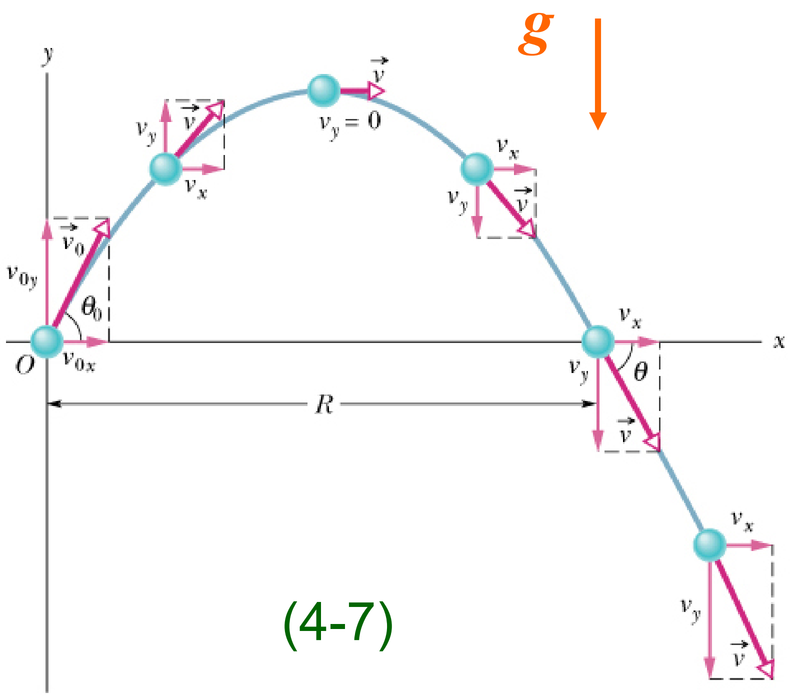
Projectile Motion

The motion of an object in a vertical plane under the influence of gravitational force is known as "projectile motion"

The projectile is launched with an initial velocity $\vec{v}_o$

The horizontal and vertical velocity components are:

$$v_{ox} = v_o \cos\theta_o \qquad v_{oy} = v_o \sin\theta_o$$



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Projectile motion will be analyzed in a horizontal and a vertical motion along the x- and y-axes, respectively. These two motions are independent of each other. Motion along the x-axis has zero acceleration. Motion along the y-axis has uniform acceleration $a_y = -g$

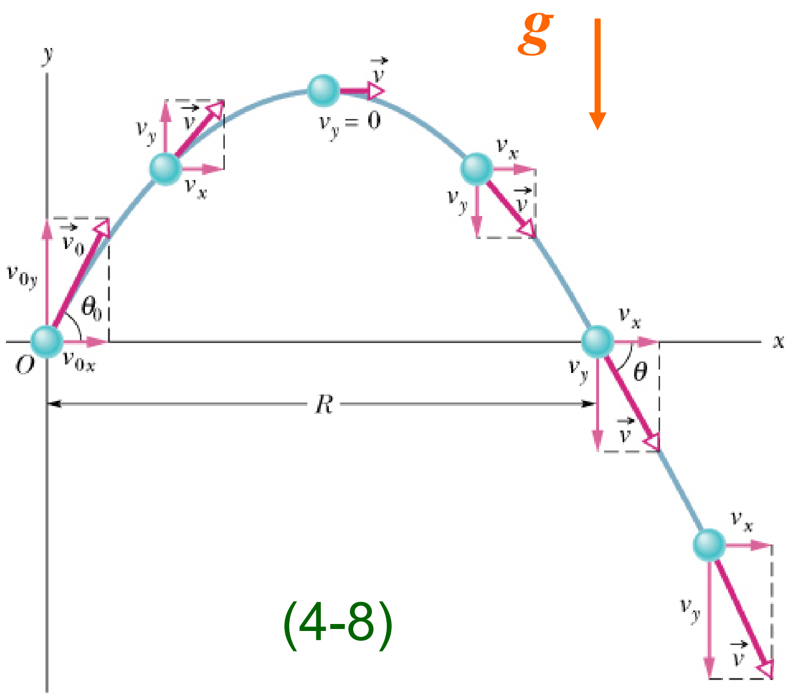
Horizontal Motion: $a_x = 0$ The velocity along the x-axis does not change

$v_x = v_0 \cos\theta_0$ (eqs.1) $x = x_o + (v_o \cos\theta_o)t$ (eqs.2)

Vertical Motion: $a_y = -g$ Along the y-axis the projectile is in free fall

$v_y = v_0 \sin\theta_0 - gt$ (eqs.3) $y = y_o + (v_0 \sin\theta_0)t - \frac{gt^2}{2}$ (eqs.4)

If we eliminate t between equations 3 and 4 we get: $v_y^2 - (v_0 \sin\theta_0)^2 = -2g(y - y_o)$



Here x_o and y_o are the coordinates of the launching point. For many problems the launching point is taken at the origin. In this case

$x_o = 0$ and $y_o = 0$

Note: In this analysis of projectile motion we neglect the effects of air resistance

The equation of the path:

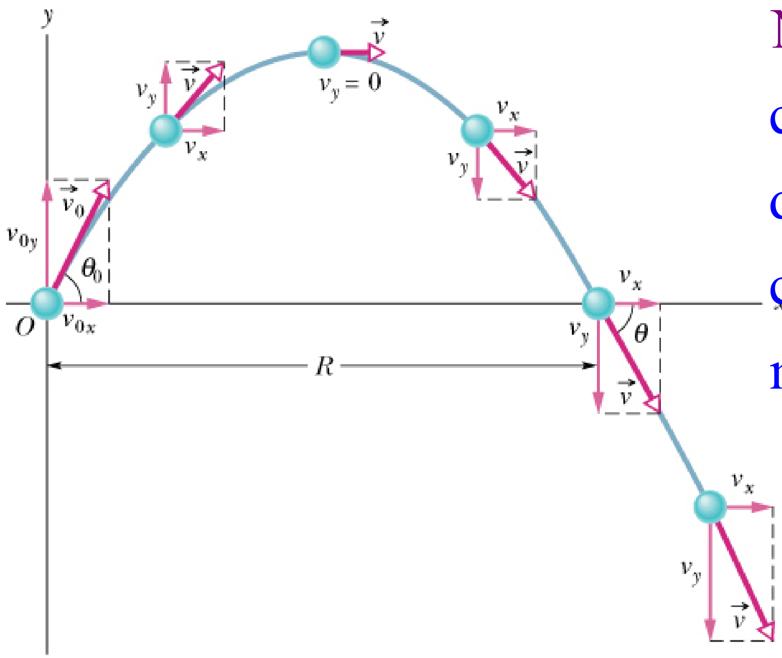
$$x = (v_o \cos\theta_o)t \quad \text{(eqs.2)} \qquad\qquad y = (v_0 \sin\theta_0)t - \frac{gt^2}{2} \quad \text{(eqs.4)}$$

If we eliminate t between equations 2 and 4 we get:

$$y = (\tan\theta_o)x - \frac{g}{2(v_o \cos\theta_o)^2}x^2$$

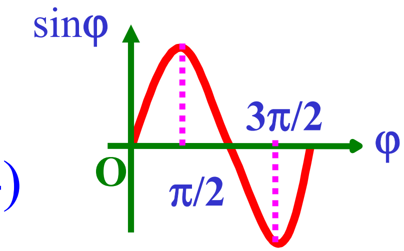
This equation describes the path of the motion

The path equations has the form: $y = ax + bx^2$ This is the equation of a parabola



Note: The equation of the path seems too complicated to be useful. Appearances can deceive: Complicated as it is, this equation can be used as a short cut in many projectile motion problems

$$v_x = v_0 \cos\theta_0 \quad \text{(eqs.1)} \qquad x = (v_o \cos\theta_o)\,t \quad \text{(eqs.2)}$$



$$v_y = v_0 \sin\theta_0 - gt \quad \text{(eqs.3)} \qquad y = (v_0 \sin\theta_0)\,t - \frac{gt^2}{2} \quad \text{(eqs.4)}$$

Horizontal Range: The distance OA is defined as the horizantal range R

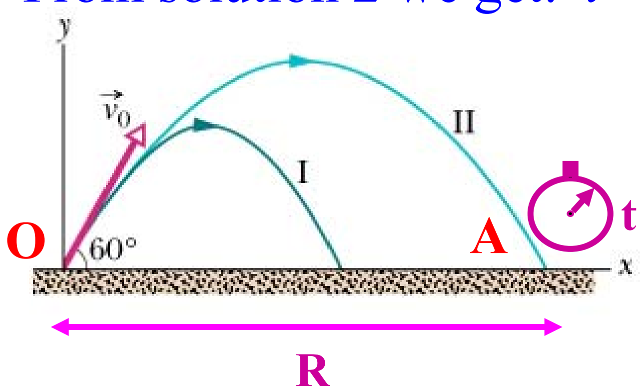
At point A we have: $y = 0$ From equation 4 we have:

$$(v_0 \sin\theta_0)\,t - \frac{gt^2}{2} = 0 \rightarrow t\left(v_0 \sin\theta_0 - \frac{gt}{2}\right) = 0$$ This equation has two solutions:

Solution 1. $t = 0$ This solution correspond to point O and is of no interest

Solution 2. $v_0 \sin\theta_0 - \frac{gt}{2} = 0$ This solution correspond to point A

From solution 2 we get: $t = \frac{2v_0 \sin\theta_0}{g}$ If we substitute t in eqs.2 we get:

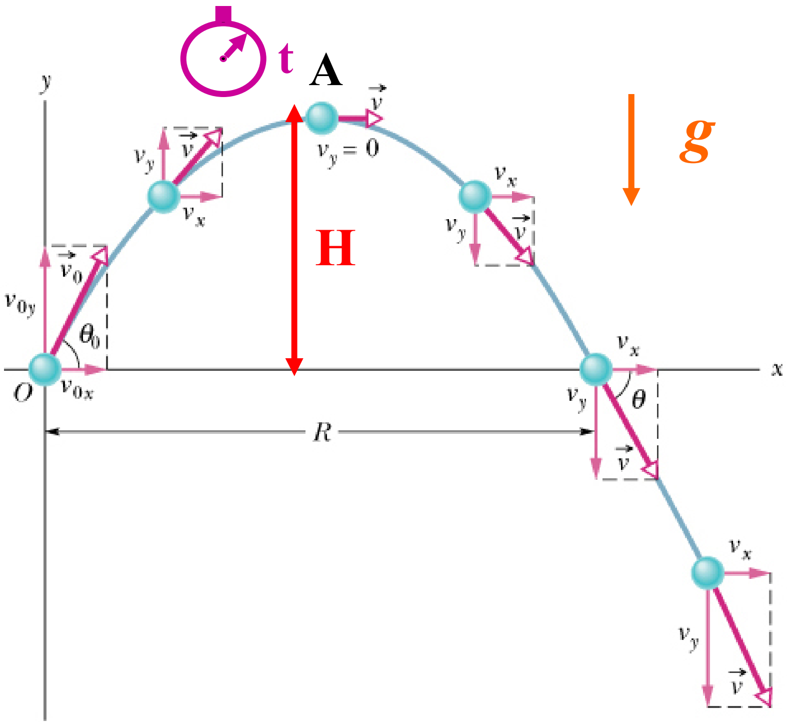


$$2\sin A\cos A = \sin 2A$$

$$R = \frac{2v_o^2}{g}\sin\theta_o \cos\theta_o = \frac{v_o^2}{g}\sin 2\theta_o$$

R has its maximum value when $\theta_o = 45°$

$$R_{max} = \frac{v_o^2}{g} \qquad (4\text{-}10)$$



Maximum height H

$$H = \frac{v_o^2 \sin^2 \theta_o}{2g}$$

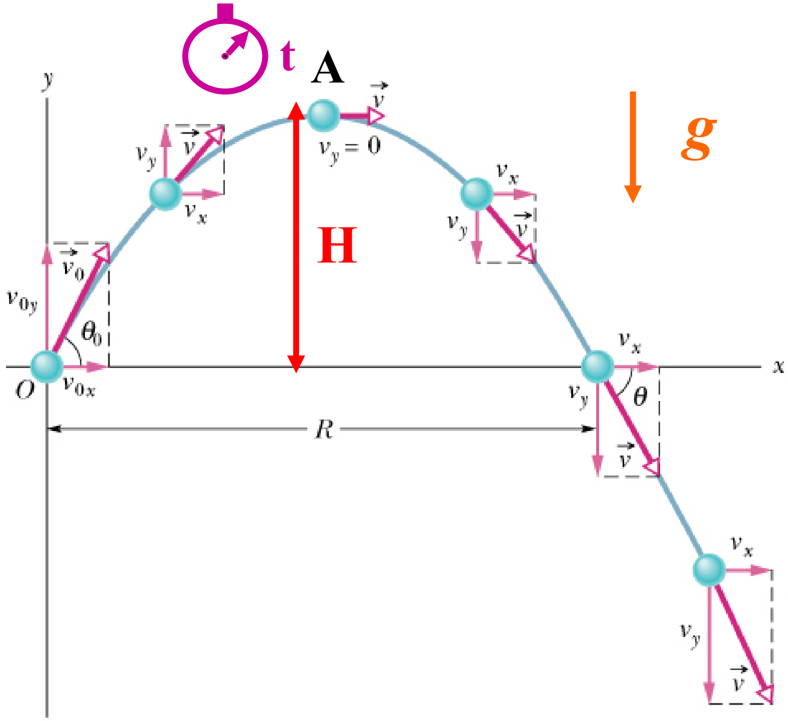
The y-component of the projectile velocity is: $v_y = v_0 \sin\theta_0 - gt$

At point A: $v_y = 0 \quad \rightarrow \; v_0 \sin\theta_0 - gt \rightarrow t = \dfrac{v_0 \sin\theta_0}{g}$

$$H = y(t) = (v_0 \sin\theta_0)t - \frac{gt^2}{2} = (v_0 \sin\theta_0)\frac{v_0 \sin\theta_0}{g} - \frac{g}{2}\left(\frac{v_0 \sin\theta_0}{g}\right)^2 \rightarrow$$

$$H = \frac{v_o^2 \sin^2 \theta_o}{2g} \qquad (4\text{-}11)$$

Maximum height H (encore)



$$H = \frac{v_o^2 \sin^2 \theta_o}{2g}$$

We can calculate the maximum height using the third equation of kinematics for motion along the y-axis: $v_y{}^2 - v_{yo}^2 = 2a(y - y_o)$

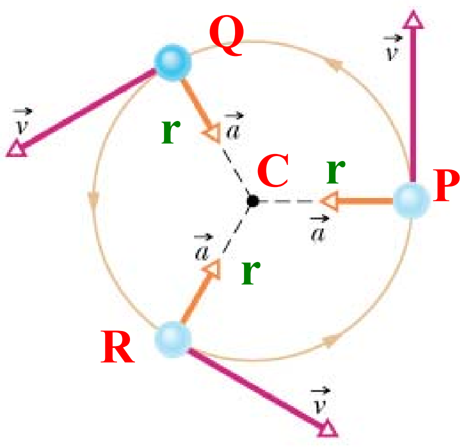
In our problem: $y_o = 0$, $y = H$, $v_{yo} = v_o \sin\theta_o$, $v_y = 0$, and $a = -g \rightarrow$

$$-v_{yo}^2 = -2gH \rightarrow H = \frac{v_{yo}^2}{2g} = \frac{v_o^2 \sin^2 \theta_o}{2g}$$

Uniform circular Motion:

A particles is in uniform circular motion it moves on a circular path of radius r with constant **speed** v. Even though the speed is constant, the velocity is not. The reason is that the direction of the velocity vector changes from point to point along the path. The fact that the velocity changes means that the acceleration is not zero. The acceleration in uniform circular motion has the following characteristics:

1. Its vector points towards the center C of the circular path, thus the name "centripetal"
2. Its magnitude a is given by the equation: $$a = \frac{v^2}{r}$$



The time T it takes to complete a full revolution is known as the "period". It is given by the equation:

$$T = \frac{2\pi r}{v}$$

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$$\vec{v} = v_x\hat{i} + v_y\hat{j} = (-v\sin\theta)\hat{i} + (v\cos\theta)\hat{j} \qquad \sin\theta = \frac{y_P}{r} \qquad \cos\theta = \frac{x_P}{r}$$

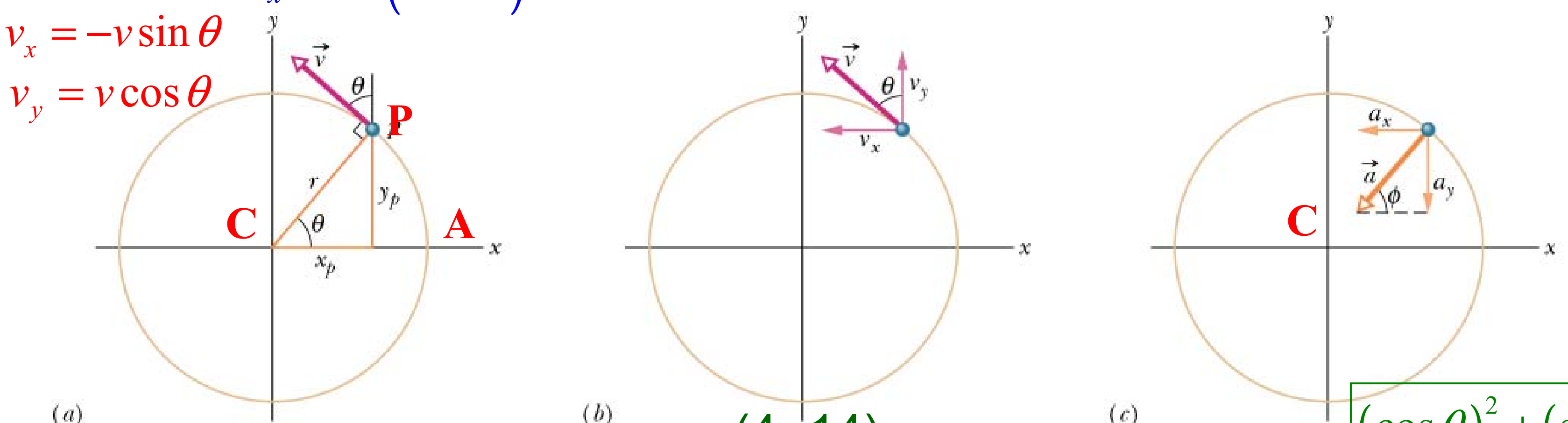
Here x_P and y_P are the coordinates of the rotating particle

$$\vec{v} = \left(-v\frac{y_P}{r}\right)\hat{i} + \left(v\frac{x_P}{r}\right)\hat{j} \qquad \text{Acceleration } \vec{a} = \frac{d\vec{v}}{dt} = \left(-\frac{v}{r}\frac{dy_P}{dt}\right)\hat{i} + \left(\frac{v}{r}\frac{dx_P}{dt}\right)\hat{j}$$

We note that: $\frac{dy_P}{dt} = v_y = v\cos\theta$ and $\frac{dx_P}{dt} = v_x = -v\sin\theta$

$$\vec{a} = \left(-\frac{v^2}{r}\cos\theta\right)\hat{i} + \left(-\frac{v^2}{r}\sin\theta\right)\hat{j} \qquad a = \sqrt{a_x^2 + a_y^2} = \frac{v^2}{r}\sqrt{(\cos\theta)^2 + (\sin\theta)^2} = \frac{v^2}{r}$$

$$\tan\phi = \frac{a_y}{a_x} = \frac{-(v^2/r)\sin\theta}{-(v^2/r)\cos\theta} = \tan\theta \to \phi = \theta \to \vec{a} \text{ points towards C}$$



$(\cos\theta)^2 + (\sin\theta)^2 = 1$

Relative Motion in One Dimension:

The velocity of a particle P determined by two different observers A and B varies from observer to observer. Below we derive what is known as the "transformation equation" of velocities. This equation gives us the exact relationship between the velocities each observer perceives. Here we assume that observer B moves with a known **constant** velocity v_{BA} with respect to observer A. Observer A and B determine the coordinates of particle P to be x_{PA} and x_{PB} , respectively.

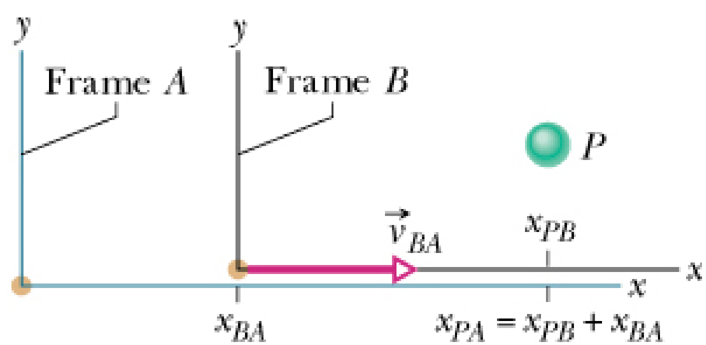
$x_{PA} = x_{PB} + x_{BA}$ Here x_{BA} is the coordinate of B with respect to A

We take derivatives of the above equation: $\frac{d}{dt}(x_{PA}) = \frac{d}{dt}(x_{PB}) + \frac{d}{dt}(x_{BA}) \rightarrow$

$$v_{PA} = v_{PB} + v_{BA}$$

If we take derivatives of the last equation and take into account that $\frac{dv_{BA}}{dt} = 0 \rightarrow$

$$a_{PA} = a_{PB}$$



Note: Even though observers A and B measure different velocities for P, they measure the same acceleration

Relative Motion in Two Dimensions:

Here we assume that observer B moves with a known **constant** velocity $\boldsymbol{v}_{BA}$ with respect to observer A in the xy-plane.

Observers A and B determine the position vector of particle P to be

$\vec{r}_{PA}$ and $\vec{r}_{PB}$, respectively.

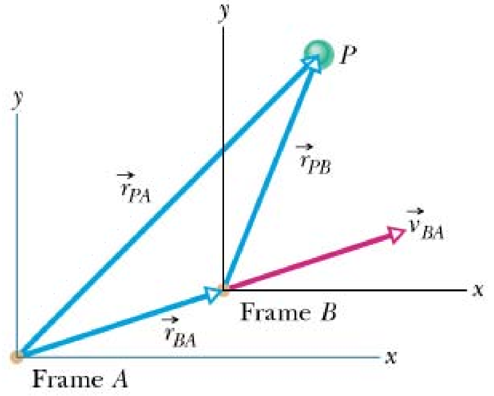
$\vec{r}_{PA} = \vec{r}_{PB} + \vec{r}_{BA}$ We take the time derivative of both sides of the equation

$$\frac{d}{dt}\vec{r}_{PA} = \frac{d}{dt}\vec{r}_{PB} + \frac{d}{dt}\vec{r}_{BA} \rightarrow \vec{v}_{PA} = \vec{v}_{PB} + \vec{v}_{BA}$$

$$\boxed{\vec{v}_{PA} = \vec{v}_{PB} + \vec{v}_{BA}}$$

If we take the time derivative of both sides of the last equation we have:

$$\frac{d}{dt}\vec{v}_{PA} = \frac{d}{dt}\vec{v}_{PB} + \frac{d}{dt}\vec{v}_{BA}$$ If we take into account that $\frac{d\vec{v}_{BA}}{dt} = 0 \rightarrow \vec{a}_{PA} = \vec{a}_{PB}$



Note: As in the one dimensional case, even though observers A and B measure different velocities for P, they measure the same acceleration