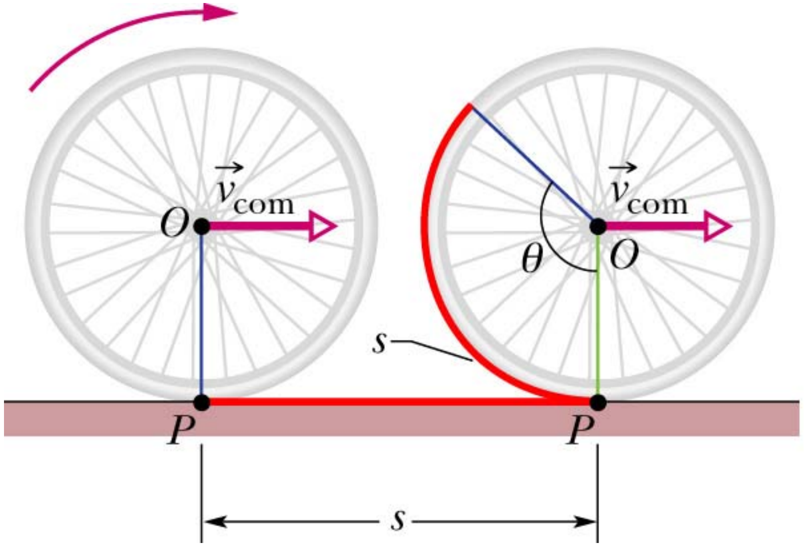
Chapter 11: Rolling, Torque, and Angular Momentum

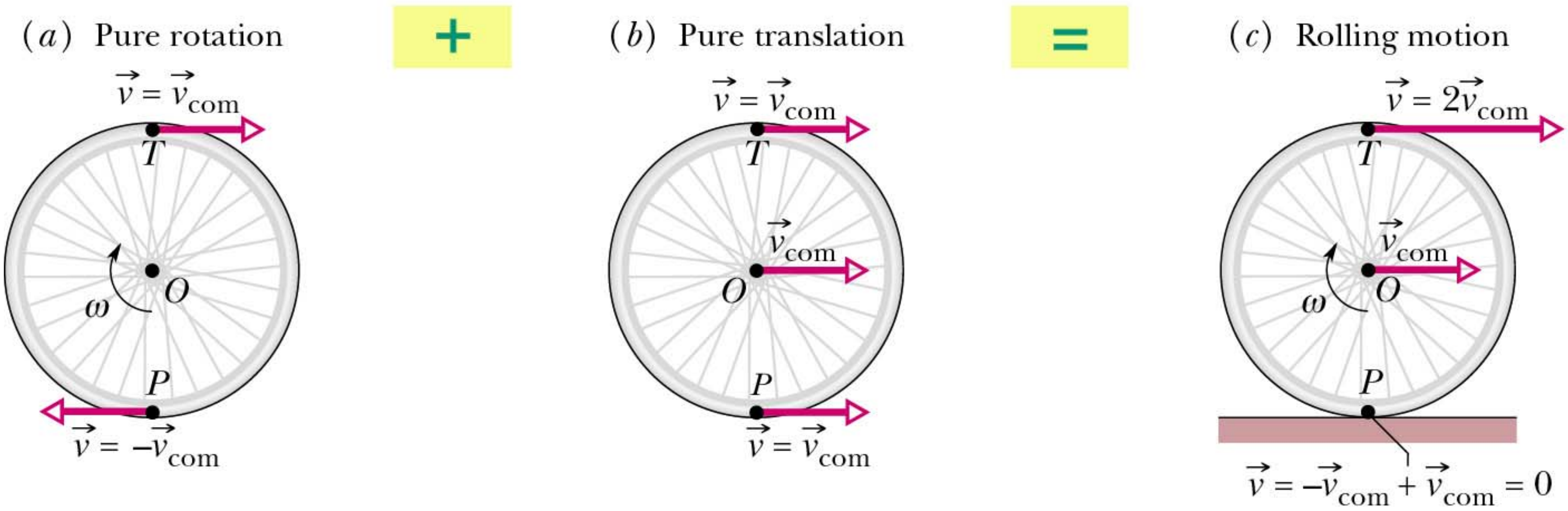
- For an object rolling smoothly, the motion of the center of mass is pure translational.

$$s = \theta R \qquad v_{com} = ds/dt = d(\theta R)/dt = \omega R$$

$$v_{com} = \omega R$$



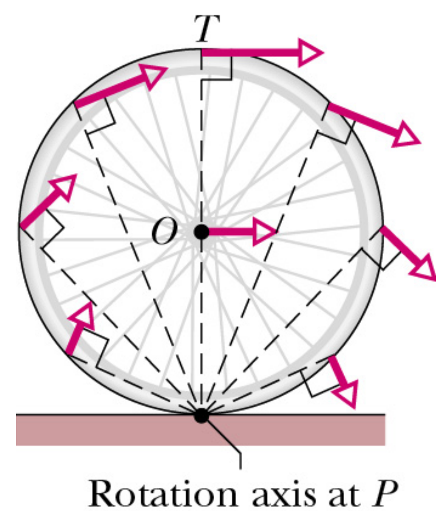
- Rolling viewed as a combination of pure rotation and pure translation



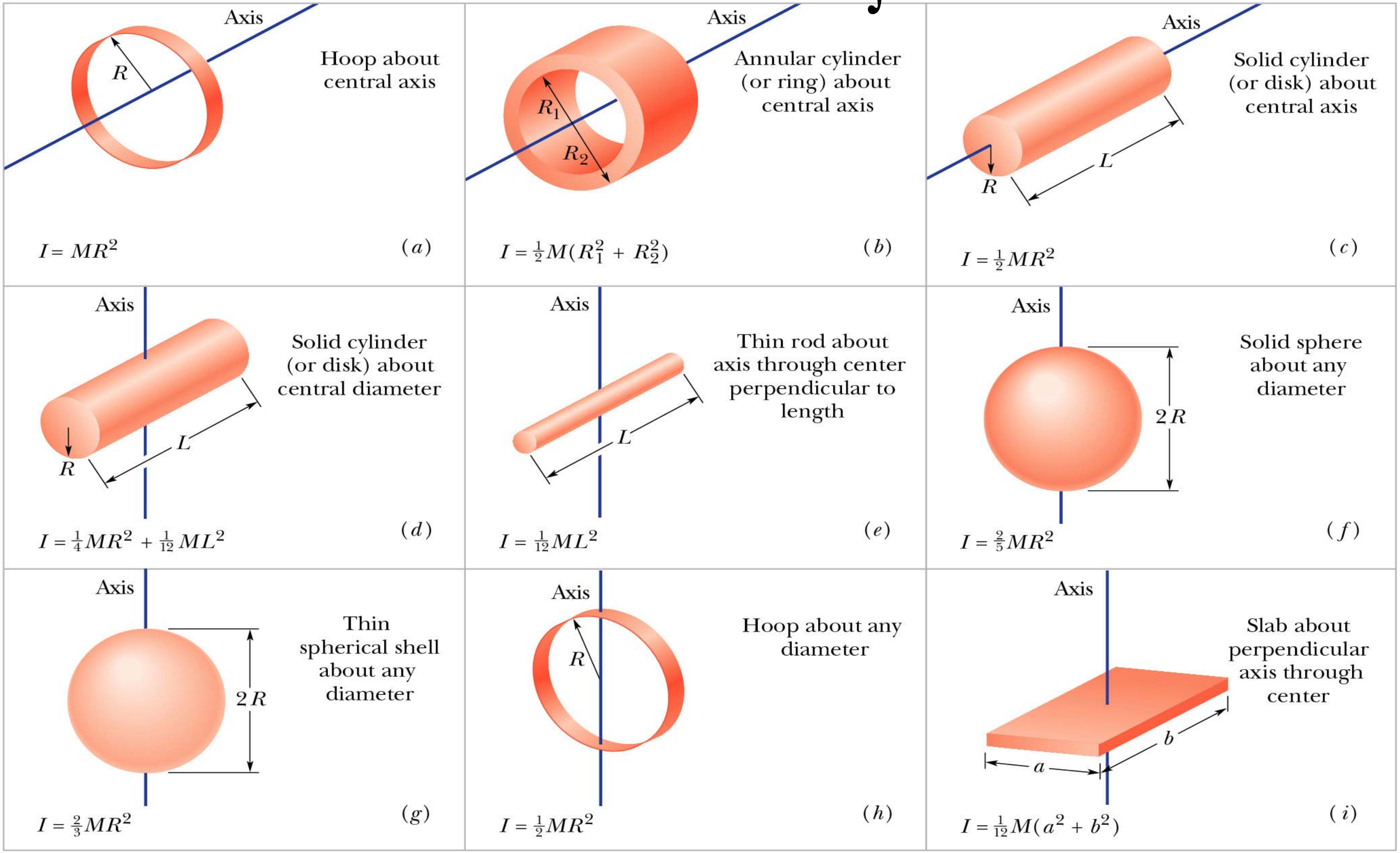
- Rolling viewed as pure rotation

$v_{top} = (\omega)(2R) = 2\ v_{com}$

- Different views, same conclusion



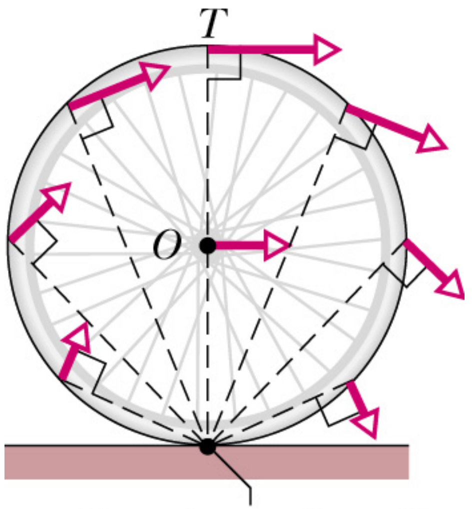
- Rotational inertia involves not only the mass but also the distribution of mass for continuous masses
- Calculating the rotational inertia $I = \int r^2 dm$



The Kinetic Energy of Rolling

View the rolling as pure rotation around P, the kinetic energy

$$K = \frac{1}{2} I_P \omega^2$$



parallel axis theorem: $I_p = I_{com} + MR^2$

so $\quad K = \frac{1}{2} I_{com}\omega^2 + \frac{1}{2} MR^2\omega^2$

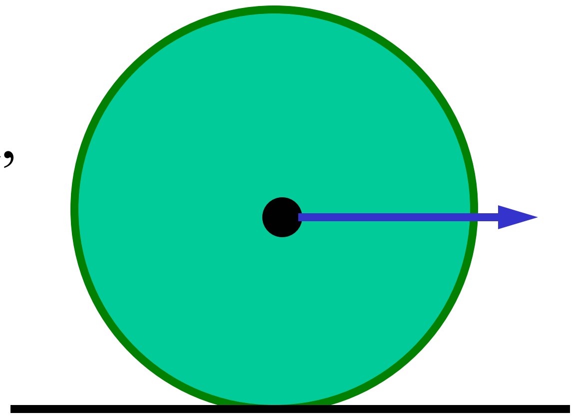
since $\quad v_{com} = \omega R$

$$\mathbf{K = \frac{1}{2} I_{com}\omega^2 + \frac{1}{2} M(v_{com})^2}$$

$\frac{1}{2} I_{com}\omega^2$: due to the object's rotation about its center of mass

$\frac{1}{2} M(v_{com})^2$: due to the translational motion of its center of mass

Sample Problem: A uniform solid cylindrical disk, of mass M = 1.4 kg and radius R = 8.5 cm, rolls smoothly across a horizontal table at a speed of 15 cm/s. What is its kinetic energy K?



$$v_{c.m.} = 0.15\text{m/s}$$

$$I_{disk} = 1/2MR^2 = (0.5)(1.4\text{kg})(0.085\text{m})^2 = 5.058\text{x}10^{-3}\text{kg m}^2$$

$$\omega = v/R = (0.15\text{m/s})/0.085\text{m} = 1.765 \text{ rad/s}$$

$$K = K_{trans} + K_{rot} = \frac{1}{2}Mv_{c.m.}^2 + \frac{1}{2}I\omega^2$$

$$K = K_{trans} + K_{rot} = \frac{1}{2}(1.4)(0.15)^2 + \frac{1}{2}(5.058\text{x}10^{-3})(1.765)^2$$

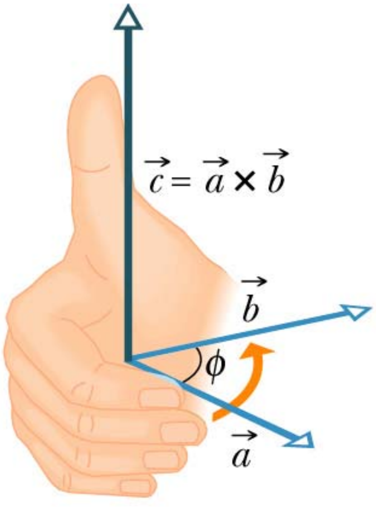
$$= 15.75\text{x}10^{-3} + 7.878\text{x}10^{-3} = 23.63\text{x}10^{-3}\text{ J}$$

Vector Product (Review)

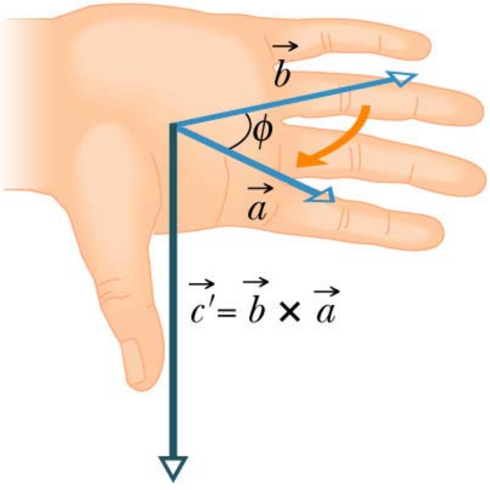
- Vector product of vectors a and b produce a third vector c whose magnitude is

  $$c = a\, b \sin \phi$$

  whose direction follow the right hand rule



(a)



(b)

if a and b parallel => a x b = 0

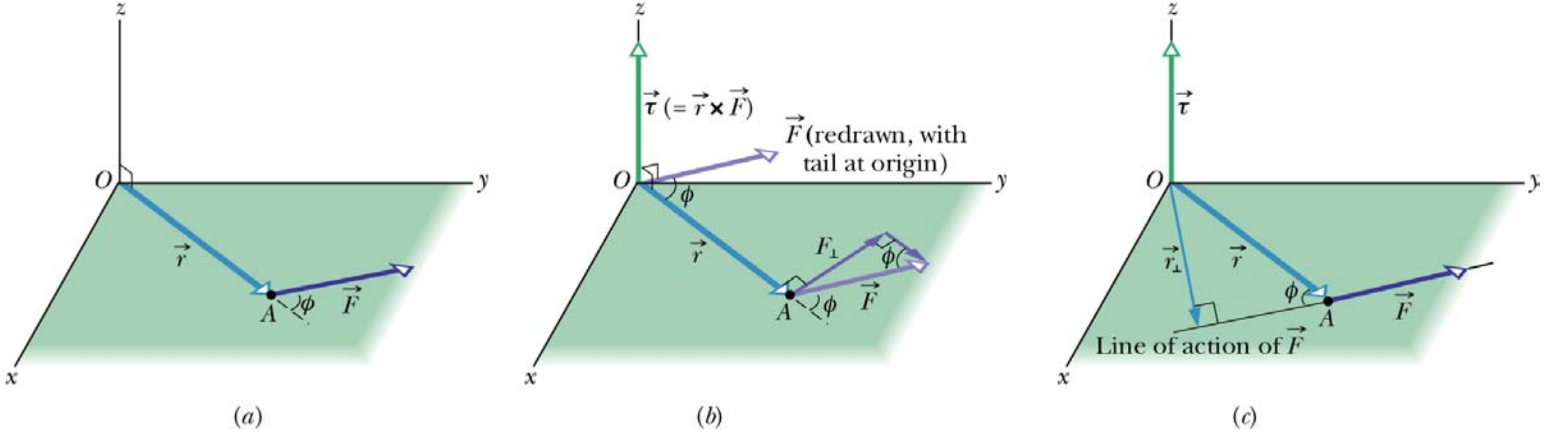
if a and b perpendicular => a x b =ab

i x i = 0 i x j = 1

Note: a x b = - (b x a)

Torque revisited

- For a fixed axis rotation, torque $\tau = r F \sin\theta$
- Expand the definition to apply to a particle that moves along any path relative to a fixed point.



$$\vec{\tau} = \vec{r} \times \vec{F}$$

- direction : right-hand rule
- magnitude :$\tau = rF \sin\Phi = rF_{\perp} = r_{\perp}F$

Angular momentum

- Angular momentum with respect to point O for a particle of mass *m* and linear momentum *p* is defined as

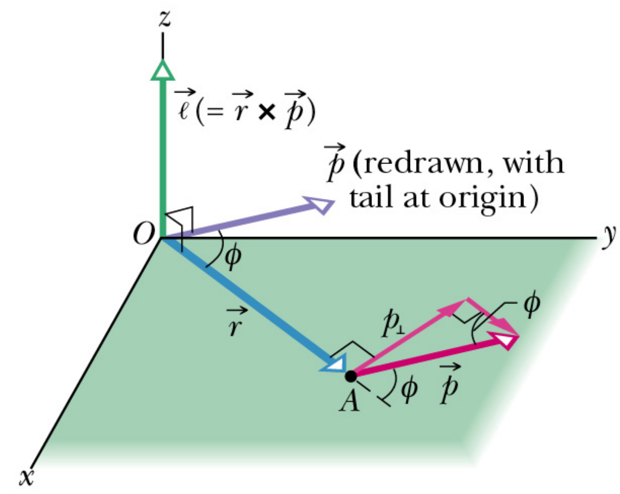
$$\vec{\ell} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$$

direction: right-hand rule

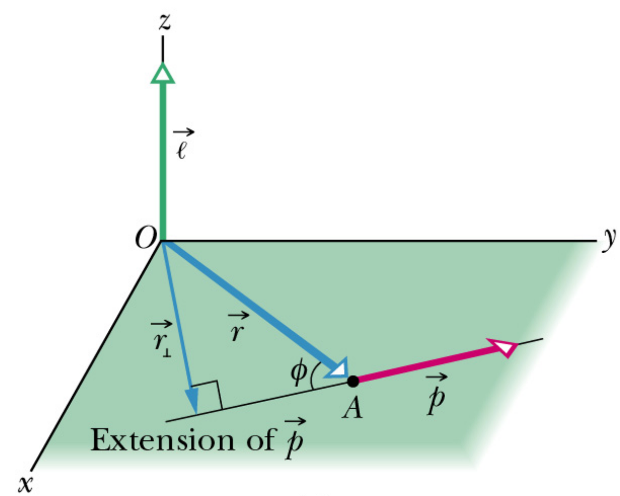
magnitude:

$$\ell = r\,p\sin\phi = r\,mv\sin\phi$$

- Compare to the linear case $\vec{p} = m\vec{v}$

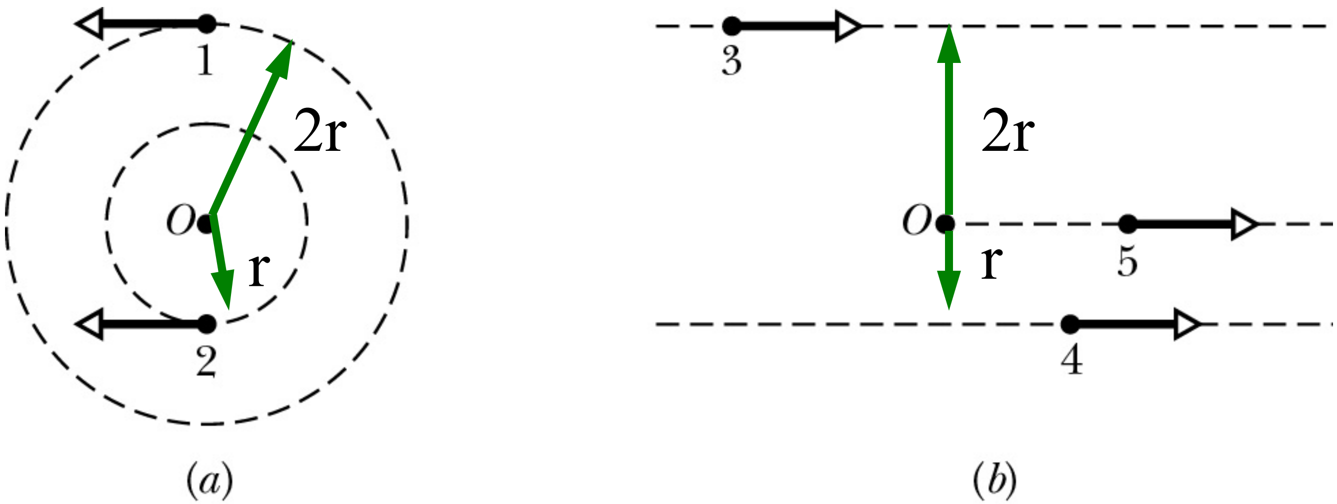


(a)



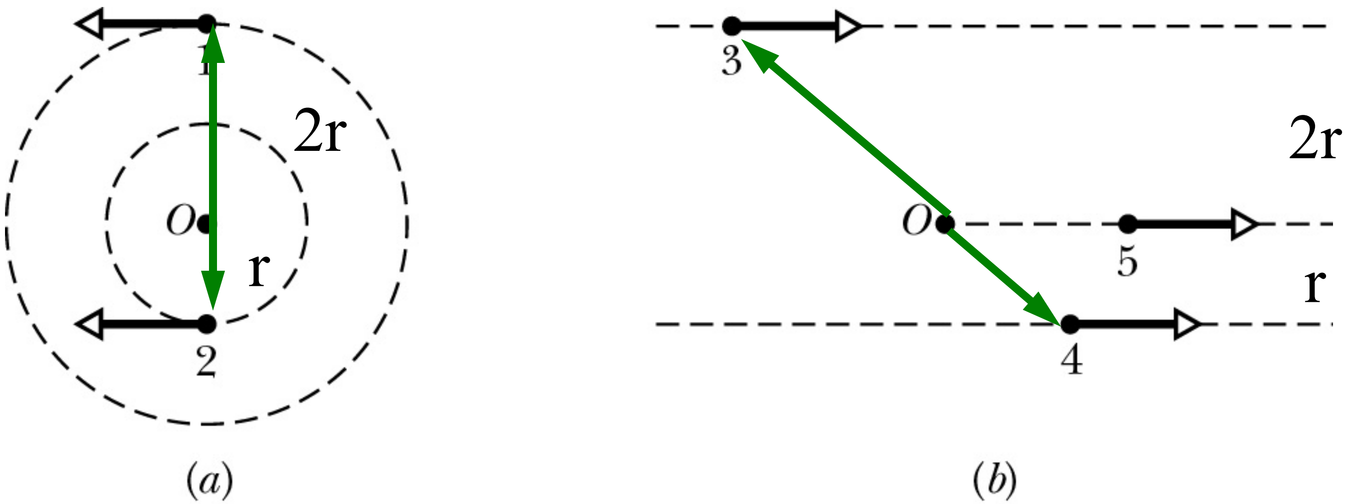
(b)

Particles 1, 2, 3, 4, 5 have the same mass and speed as shown in the figure. Particles 1 & 2 move around O in opposite directions. Particles 3, 4, and 5 move towards or away from O as shown.



Which of the particles has the greatest magnitude angular momentum?

Particles 1, 2, 3, 4, 5 have the same mass and speed as shown in the figure. Particles 1 & 2 move around *O* in opposite directions. Particles 3, 4, and 5 move towards or away from *O* as shown.



$\ell = r\,p\sin\phi = r\,mv\sin\phi$ $\phi = 90^{o}$ only for 1 and 2, but $r_1 = 2r_2$.

Which of the particles has the greatest magnitude angular momentum?

1) 1 2) 2 3) 3 4) 4 5) 5 6) all have the same *l*

Newton's Second Law in Angular Form

$$\vec{\tau}_{net} = \frac{d\vec{l}}{dt}$$

- $\vec{\tau}_{net}$: the vector sum of all the torques acting on the object
- Comparing to the linear case: $\vec{F}_{net} = \frac{d\vec{p}}{dt}$
- Newton's 2nd law for a system of particles

$$\vec{\tau}_{net} = \frac{d\vec{L}_{total}}{dt} \qquad \vec{L}_{total} = \sum_{i=1}^{n} \vec{l}_i$$

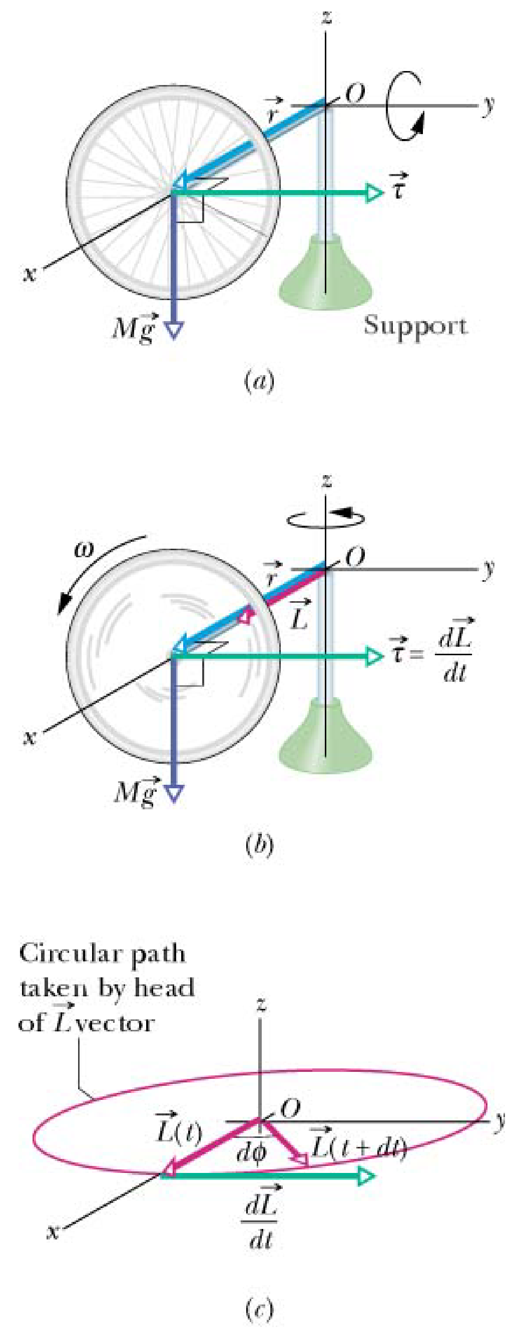
  – Net external torque equals to the time rate change of the system's total angular momentum

Torque and Angular Momentum

$$\vec{\tau}_{net} = \frac{d\vec{l}}{dt} = \frac{d(\vec{r}\times m\vec{v})}{dt}$$

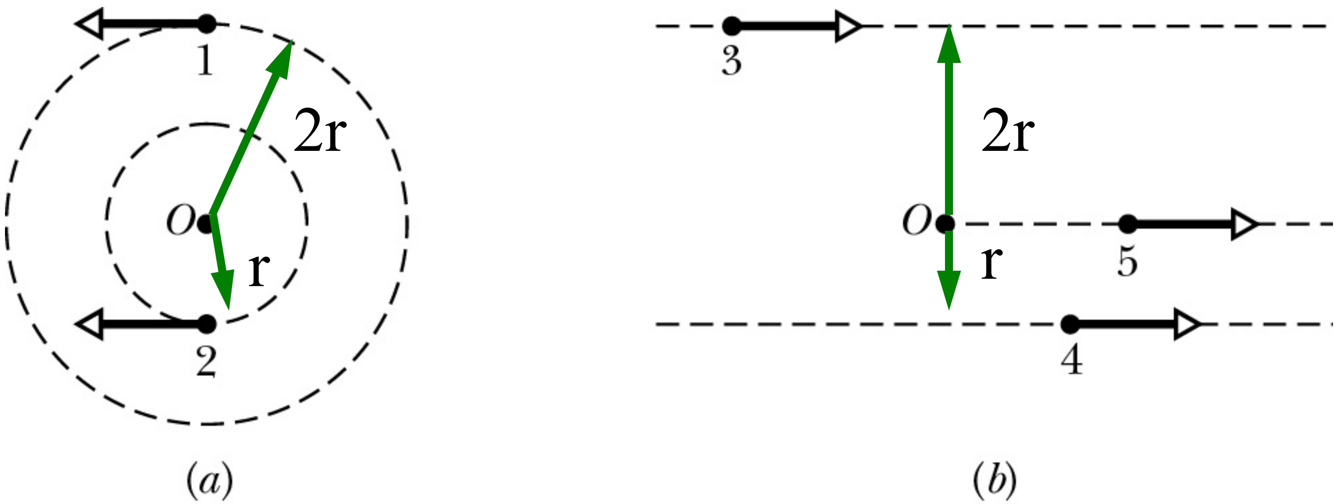
$$= \vec{r}\times\left(m\frac{d\vec{v}}{dt}\right) = \vec{r}\times(m\vec{a}) = \vec{r}\times\vec{F}$$

Torque is the time rate of change of angular momentum.



A Quiz

Particles 1, 2, 3, 4, 5 have the same mass and speed as shown in the figure. Particles 1 & 2 move around *O* in opposite directions. Particles 3, 4, and 5 move towards or away from *O* as shown.

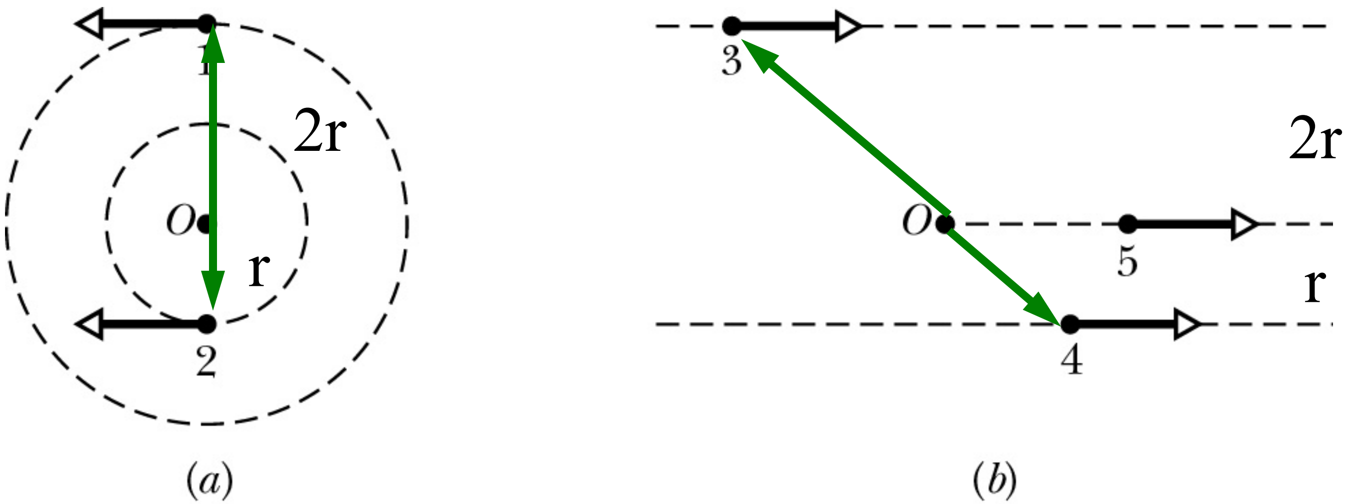


Which of the particles has the smallest magnitude angular momentum?

1) 1 2) 2 3) 3 4) 4 5) 5 6) all have the same l

A Quiz

Particles 1, 2, 3, 4, 5 have the same mass and speed as shown in the figure. Particles 1 & 2 move around *O* in opposite directions. Particles 3, 4, and 5 move towards or away from *O* as shown.



$$l = r\,p\sin\phi = r\,m v\sin\phi$$

$\phi = 0^o$ for 5. => $l = 0$

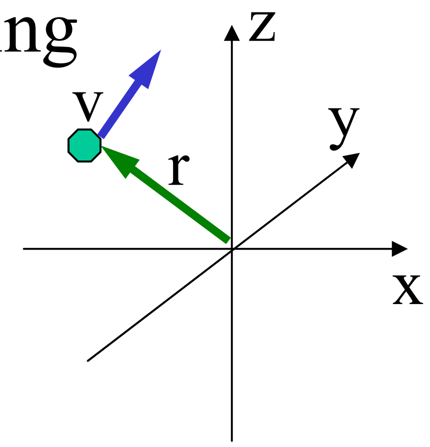
Which of the particles has the smallest magnitude angular momentum?

1) 1 2) 2 3) 3 4) 4 5) 5 6) all have the same l

The angular momentum of a rigid body rotating about a fixed axis

- Consider a simple case, a mass *m* rotating about a fixed axis *z*:

  $l = r\ mv\ \sin 90^o = r\ m\ r\ \omega = mr^2\omega = I\ \omega$



- In general, the angular momentum of rigid body rotating about a fixed axis is

$$L = I\ \omega$$

L : angular momentum along the rotation axis

I : moment of inertia about the same axis

Conservation of Angular Momentum

- If the net external torque acting on a system is zero, the angular momentum of the system is conserved.

$$\text{if} \quad \vec{\tau}_{net} = \frac{d\vec{L}}{dt} \qquad \vec{\tau}_{net} = 0 \qquad \text{then} \quad \vec{L} = const$$

- If $\tau_{\text{net, z}} = 0$ then $L_{i,\,z} = L_{f,\,z}$ $\qquad \vec{L}_i = \vec{L}_f$

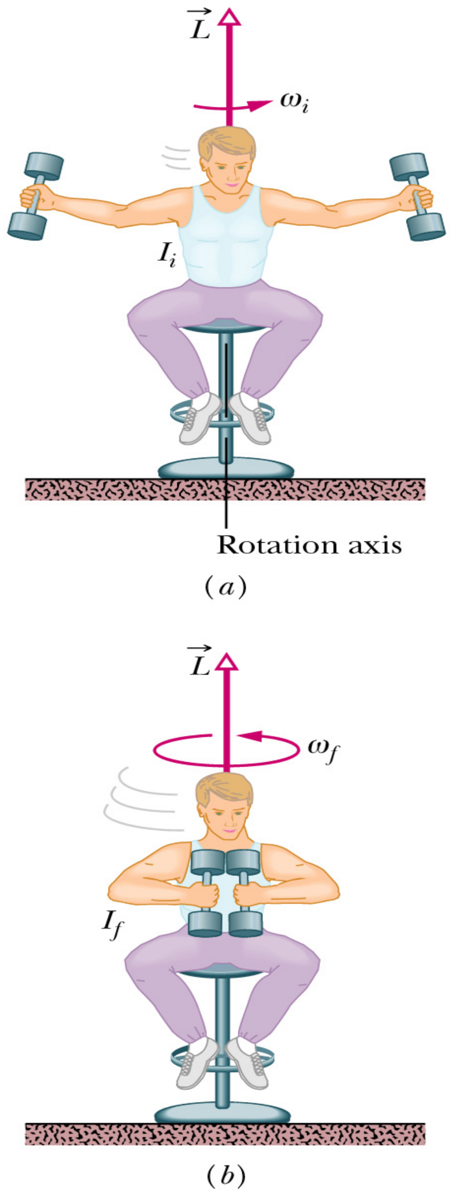
- For a rigid body rotating around a fixed axis, ($L = I\,w$) the conservation of angular momentum can be written as

$$I_i\,\omega_i = I_f\,\omega_f$$

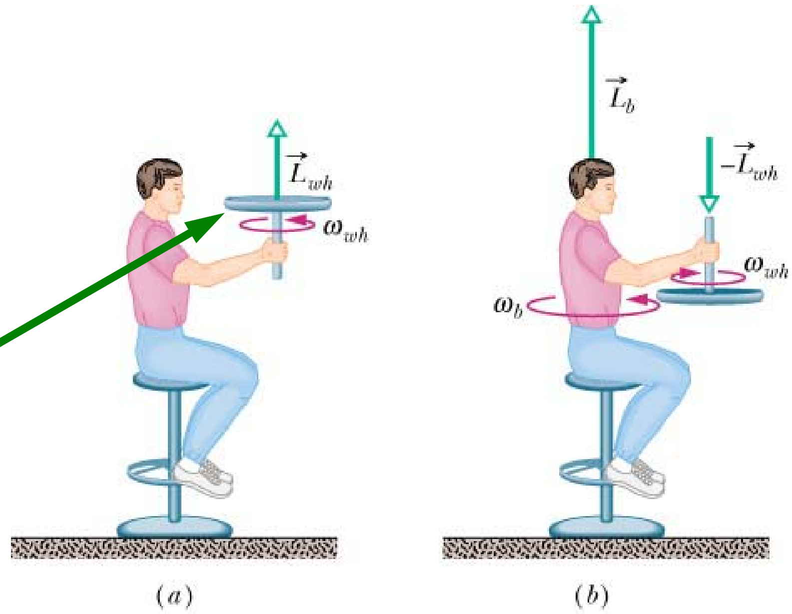
Some examples involving conservation of angular momentum

- The spinning volunteer

$$I_i \omega_i = I_f \omega_f$$

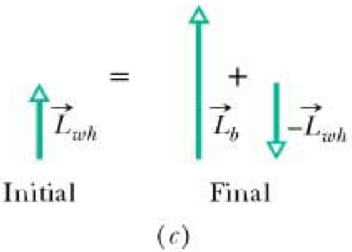


Angular momentum is conserved



L_i is in the spinning wheel

Now exert a torque to flip its rotation.
$L_{f,\text{ wheel}} = -L_i$.

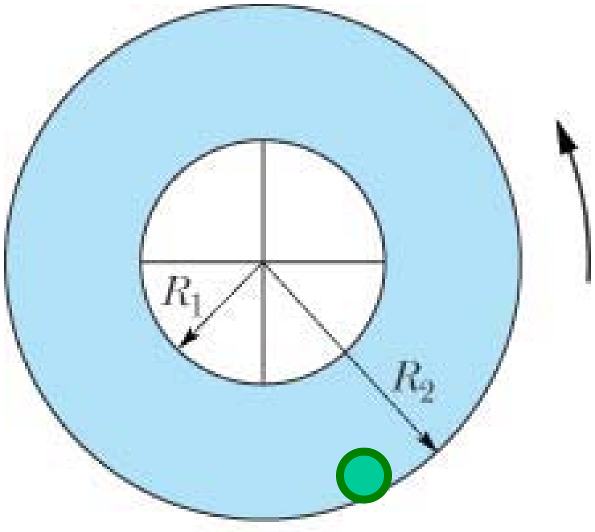


Conservation of Angular momentum means that the person must now acquire an angular momentum.

$L_{f,\text{ person}} = +2L_i$

so that $L_f = L_{f,\text{ person}} + L_{f,\text{ wheel}} = +2L_i + -L_i = L_i$.

Problem 11-66



Ring of R_1 (=$R_2/2$) and R_2 (=0.8m),
Mass m_2 = 8.00kg.
ω_i = 8.00 rad/s. Cat m_1 = 2kg. Find kinetic energy change when cat walks from outer radius to inner radius.

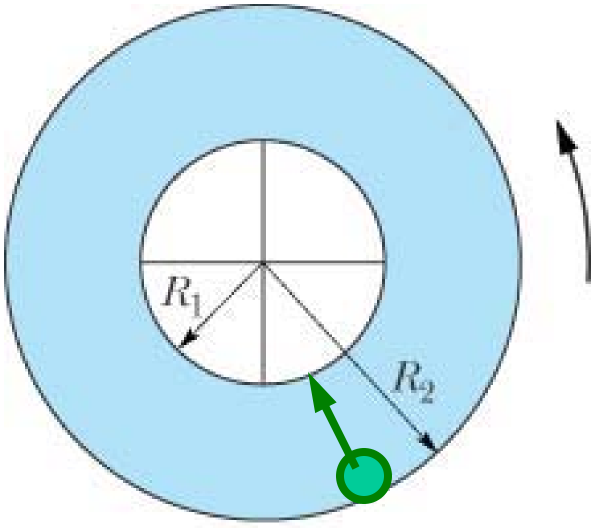
Initial Momentum

$$L_i = L_{i,cat} + L_{i,ring} = m_1 R_2 v_i + I\omega_i$$

$$= m_1 R_2^2 \omega_i + \frac{1}{2} m_2 (R_1^2 + R_2^2)\omega_i$$

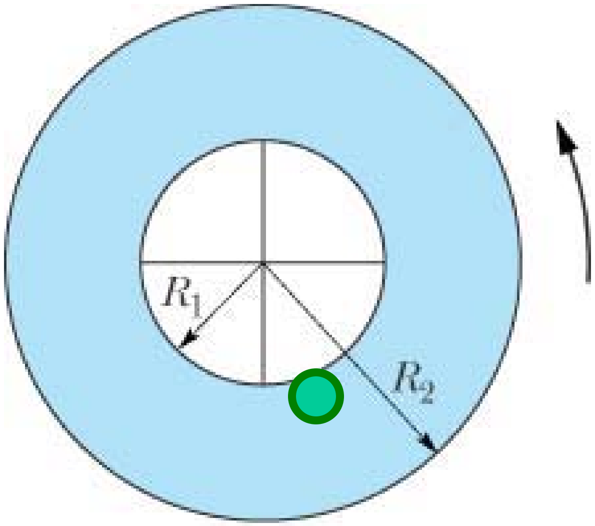
$$= m_1 R_2^2 \omega_i \left(1 + \frac{1}{2}\frac{m_2}{m_1}\left(\frac{R_1^2}{R_2^2} + 1\right)\right)$$

Problem 11-66



Ring of R_1 (=R_2/2) and R_2 (=0.8m),
Mass m_2 = 8.00kg.
ω_i = 8.00 rad/s. Cat m_1 = 2kg. Find kinetic energy change when cat walks from outer radius to inner radius.

Problem 11-66



Ring of R_1 (=R_2/2) and R_2 (=0.8m),
Mass m_2 = 8.00kg.
ω_i = 8.00 rad/s. Cat m_1 = 2kg. Find kinetic energy change when cat walks from outer radius to inner radius.

Final Momentum

$$L_f = L_{f,cat} + L_{f,ring} = m_1 R_1 v_f + I\omega_f$$

$$= m_1 R_1^2 \omega_f + \frac{1}{2} m_2 (R_1^2 + R_2^2)\omega_f$$

$$= m_1 R_1^2 \omega_f \left(1 + \frac{1}{2} \frac{m_2}{m_1} \left(\frac{R_2^2}{R_1^2} + 1 \right) \right)$$

Problem 11-66

Then from $L_f = L_i$ we obtain

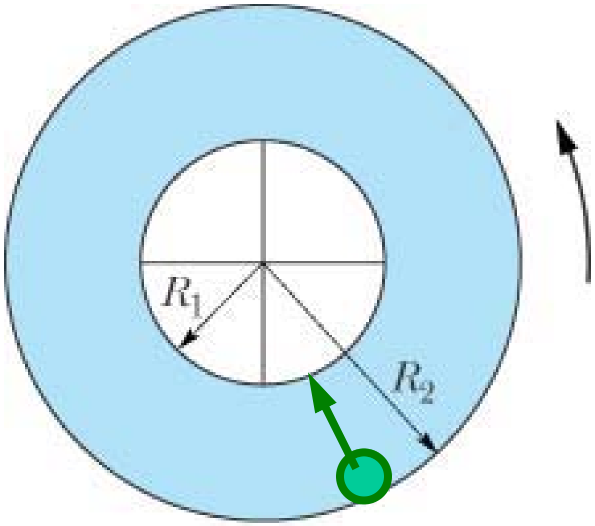
$$\frac{\omega_f}{\omega_0} = \frac{R_2^2}{R_1^2} \frac{1+\frac{1}{2}\frac{m_2}{m_1}\left(\frac{R_1^2}{R_2^2}+1\right)}{1+\frac{1}{2}\frac{m_2}{m_1}\left(1+\frac{R_2^2}{R_1^2}\right)} = (2.0)^2 \frac{1+2(0.25+1)}{1+2(1+4)} = 1.273$$

Thus, $\omega_f = 1.273\omega_0$. Using $\omega_0 = 8.00$ rad/s, we have $\omega_f = 10.2$ rad/s. By substituting $I = L/\omega$ into $K = \frac{1}{2}I\omega^2$, we obtain $K = \frac{1}{2}L\omega$. Since $L_i = L_f$, the kinetic energy ratio becomes

$$\frac{K_f}{K_i} = \frac{\frac{1}{2}L_f\omega_f}{\frac{1}{2}L_i\omega_i} = \frac{\omega_f}{\omega_0} = 1.273.$$

which implies $\Delta K = K_f - K_i = 0.273K_i$. The cat does positive work while walking toward the center of the ring, increasing the total kinetic energy of the system.

Problem 11-66



Ring of R_1 (=R_2/2) and R_2 (=0.8m),
Mass m_2 = 8.00kg.
ω_i = 8.00 rad/s. Cat m_1 = 2kg. Find kinetic energy change when cat walks from outer radius to inner radius.

Initial Kinetic energy K_i is:

$$K_i = \frac{1}{2}\left[m_1 R_2^2 + \frac{1}{2} m_2 (R_1^2 + R_2^2)\right]\omega_0^2 = \frac{1}{2} m_1 R_2^2 \omega_0^2 \left[1 + \frac{1}{2}\frac{m_2}{m_1}\left(\frac{R_1^2}{R_2^2} + 1\right)\right]$$

$$= \frac{1}{2}(2.00\text{ kg})(0.800\text{ m})^2(8.00\text{ rad/s})^2[1+(1/2)(4)(0.5^2+1)]$$

$$= 143.36\text{ J},$$

the increase in kinetic energy is $\Delta K = (0.273)(143.36\text{ J}) = 39.1\text{ J}.$