Chapter 10: Rotation

- Review of translational motion (motion along a straight line)
  - Position x
  - Displacement Δx
  - Velocity $v = dx/dt$
  - Acceleration $a = dv/dt$
  - Mass m
  - Newton's second law $F = ma$
  - Work $W = Fd\cos\phi$
  - Kinetic energy $K = \frac{1}{2} mv^2$

- What about rotational motion?

Rotational variables

- We will focus on the rotation of a rigid body about a fixed axis
- **Rotation axis**
- **Reference line**: pick a point, draw a line perpendicular to the rotation axis
- **Angular position**

  zero angular position

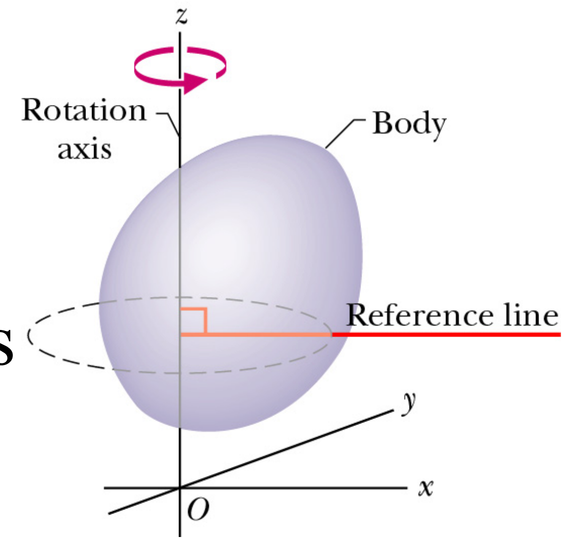
  angular position: $\theta = s/r$

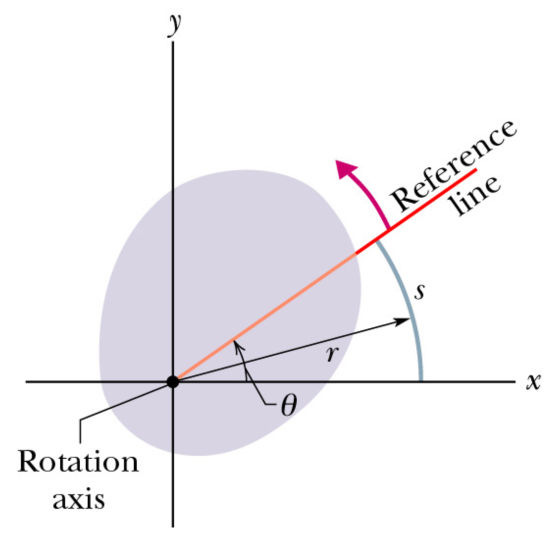
  s: length of the arc, r: radius

  Unit of θ: radians (rad)

  1 rev = $360^o = 2\pi r/r = 2\pi$ rad

  1 rad = 57.3^o = 0.159 rev

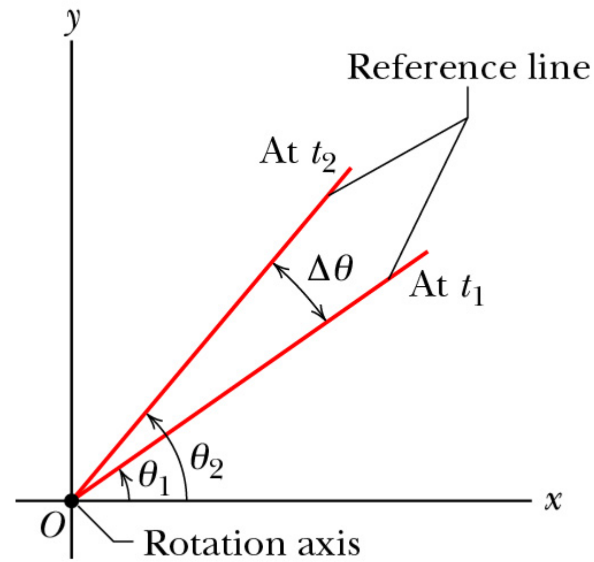




- **Angular displacement**

  $\Delta\theta = \theta_2 - \theta_1$

  direction: "clock is negative"



- **Angular velocity**

  average: $\omega_{avg} = \Delta\theta/\Delta t$

  instantaneous: $\omega = d\theta/dt$

  unit: rad/s, rev/s, direction: "clock is negative"

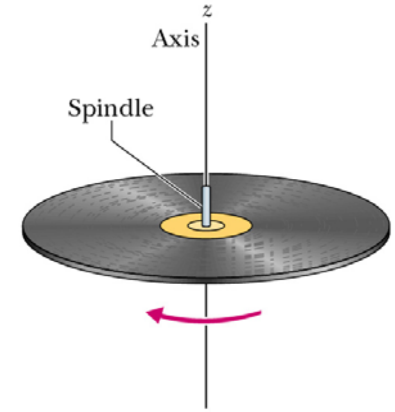
  magnitude of angular velocity = angular speed

- **Angular acceleration**

  average: $\alpha_{avg} = \Delta\omega/\Delta t$

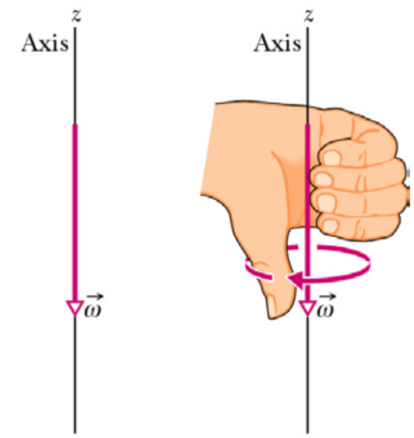
  instantaneous: $\alpha = d\omega/dt$ unit: rad/s^2

- Angular velocity and angular acceleration are vectors.
- For rotation along a fixed axis, we need not consider vectors. We can just use “+” and “-” sign to represent the direction of ω and α.
  “clock is negative”



(a)

- Direction of ω : right hand rule



(b) (c)

Rotation with constant angular acceleration

- The equations for constant angular acceleration are similar to those for constant linear acceleration

replace θ for x, ω for v, and α for a,

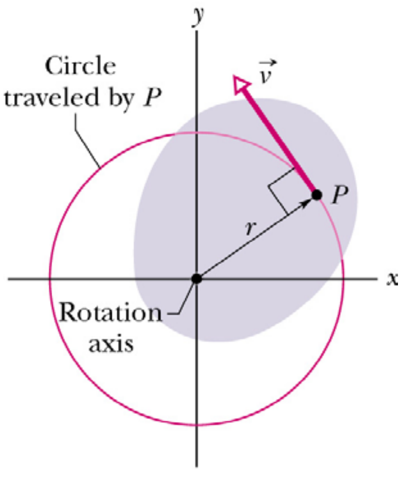
Linear		Angular	missing
$v = v_0 + at$	→	$\omega = \omega_0 + \alpha t$	$\theta - \theta_0$
$x - x_0 = v_0 t + \frac{1}{2} at^2$	→	$\theta - \theta_0 = \omega_0 t + \frac{1}{2} \alpha t^2$	ω
$v^2 = v_0^2 + 2a(x-x_0)$	→	$\omega^2 = \omega_0^2 + 2\alpha (\theta - \theta_0)$	t

and two more equations

Linear		Angular	missing
$x - x_0 = \frac{1}{2} (v_0 + v)t$	→	$\theta - \theta_0 = \frac{1}{2}(\omega_0 - \omega)t$	α
$x - x_0 = vt - \frac{1}{2} at^2$	→	$\theta - \theta_0 = \omega t - \frac{1}{2} \alpha t^2$	ω_0

Relating the linear and angular variables

- The linear and angular quantities are related by r



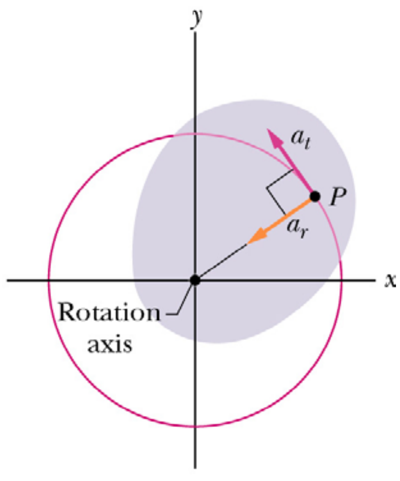
(a)

- **The position**
  - distance $s = \theta\, r$ **θ *in* radians!**
- **The speed**

  $ds/dt = d(\theta\, r)/dt = (d\theta/dt)r$

  $v = \omega\, r$
- **Time for one revolution**

  $T = 2\pi r/v = 2\pi/\omega$



(b)

- Note: θ and ω must be in radian measure

Acceleration

$dv/dt = d(\omega r)/dt = (d\omega/dt)r$

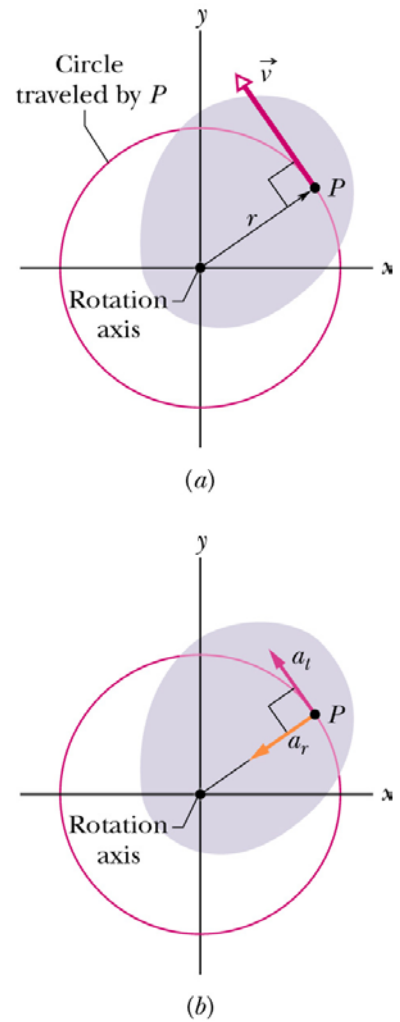
- <u>tangential component</u>

  $a_t = \alpha r \quad (\alpha = d\omega/dt)$

  α must be in radian measure

- <u>radial component</u>

  $a_r = v^2/r = (\omega r)^2/r = \omega^2 r$

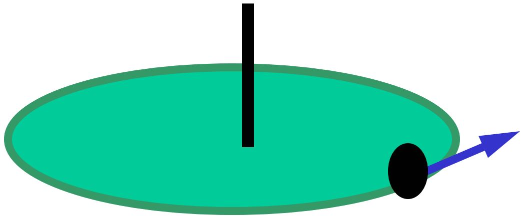


Note: a_r is present whenever angular velocity is not zero (i.e. when there is rotation), a_t is present whenever angular acceleration is not zero (i.e. the angular velocity is not constant)

Cp11-3: A cockroach rides the rim of a rotating merry-go-around. If the angular speed of this system (merry-go-around+ cockroach) is constant, does the cockroach have

(a) radial acceleration?

(b) tangential acceleration?



If the angular speed is decreasing, does the cockroach have

(c) radial acceleration ?

(d) tangential acceleration ?

Kinetic Energy of Rotation

- Consider a rigid body rotating around a fixed axis as a collection of particles with different linear speed, the total kinetic energy is

  $K = \Sigma \frac{1}{2} m_i v_i^2 = \Sigma \frac{1}{2} m_i (\omega r_i)^2 = \frac{1}{2} (\Sigma m_i r_i^2) \omega^2$

- Define **rotational inertia (moment of inertia)** to be

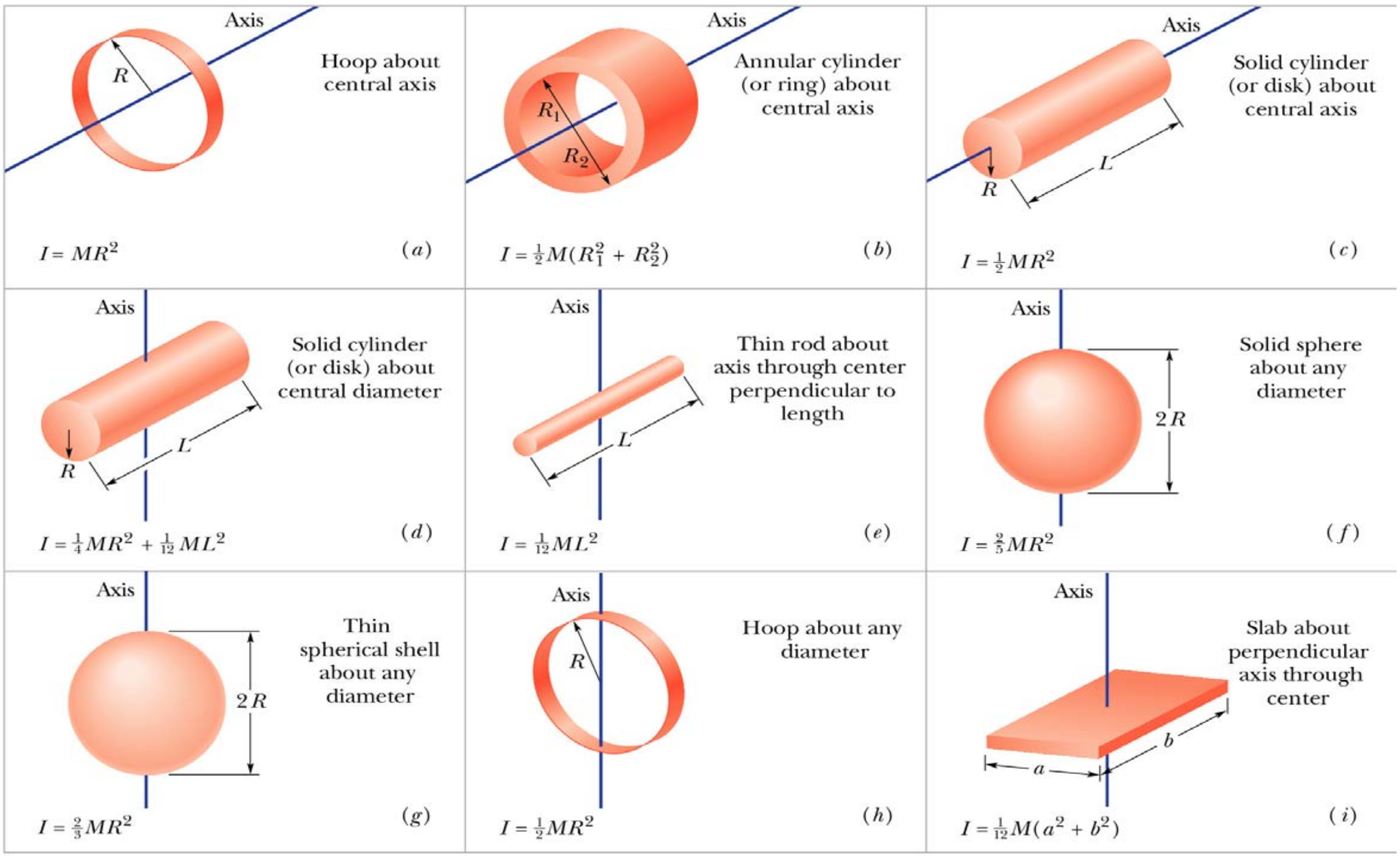
  $$I = \Sigma m_i r_i^2$$

  r_i : the perpendicular distance between m_i and the given rotation axis

- Then $K = \frac{1}{2} I \omega^2$

  Compare to the linear case: $K = \frac{1}{2} m v^2$

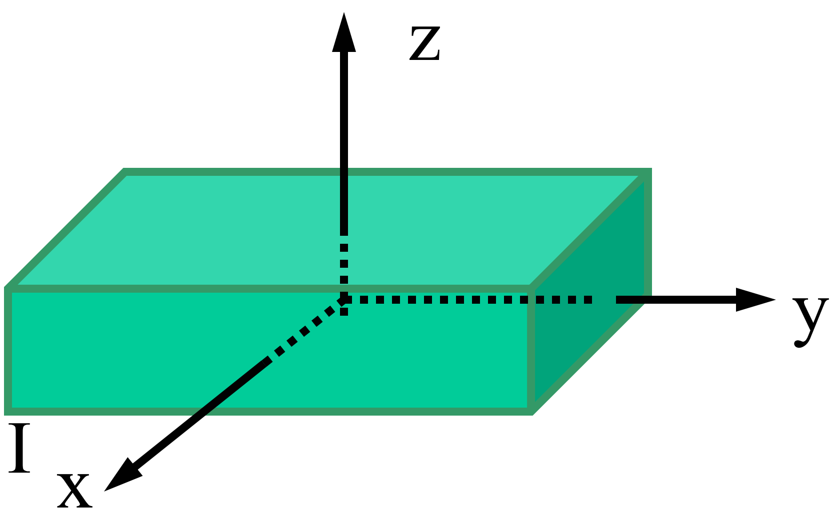
- Rotational inertia involves not only the mass but also the distribution of mass for continuous masses
- Calculating the rotational inertia $I = \int r^2 dm$



A Quiz

Consider a standard blackboard eraser. Rotation along which axis would it have the greatest moment of inertia I?

1) x
2) y
3) z
4) All have the same I



A Quiz

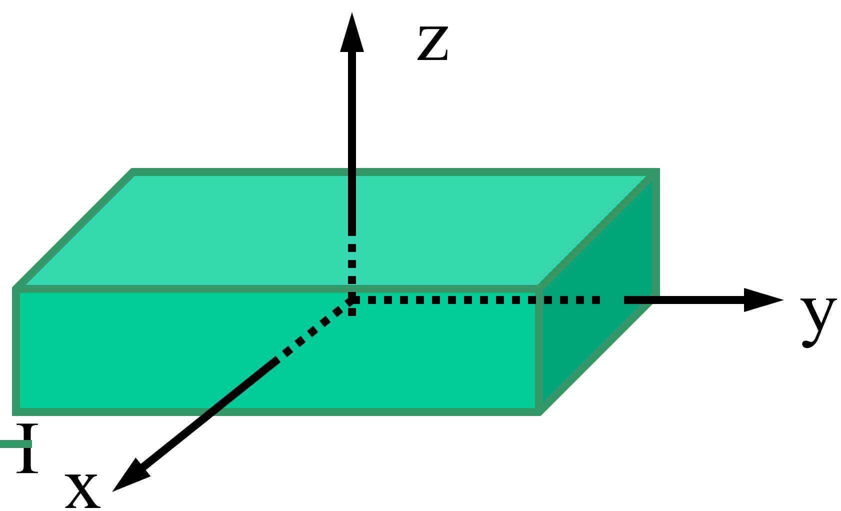
$$I = \frac{1}{12} M\left(a^2 + b^2\right)$$

Look for largest amount of mass away from the axis.

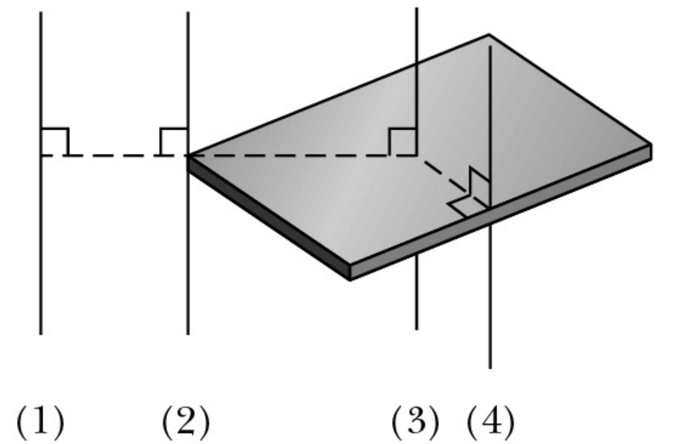
Note $\Delta y > \Delta x > \Delta z$

Consider a standard blackboard eraser. Rotation along which axis would it have the greatest moment of inertia I?

1) ~~x~~ a, b = $\Delta y, \Delta z$
2) ~~y~~ a, b = $\Delta x, \Delta z$
3) z a, b = $\Delta x, \Delta y$
4) ~~All have the same I~~



Parallel-Axis theorem



- If we know the rotational inertia of a body about any axis that passes through its **center-of-mass**, we can find its rotational inertia about any other axis **parallel** to that axis with the **parallel axis theorem**

$$I = I_{c.m.} + M\,h^2$$

h: the perpendicular distance between the two axes

Torque

- The ability of a force F to rotate a body depends not only on its magnitude, but also on its direction and where it is applied.
- **Torque** is a quantity to measure this ability Torque is a **VECTOR**

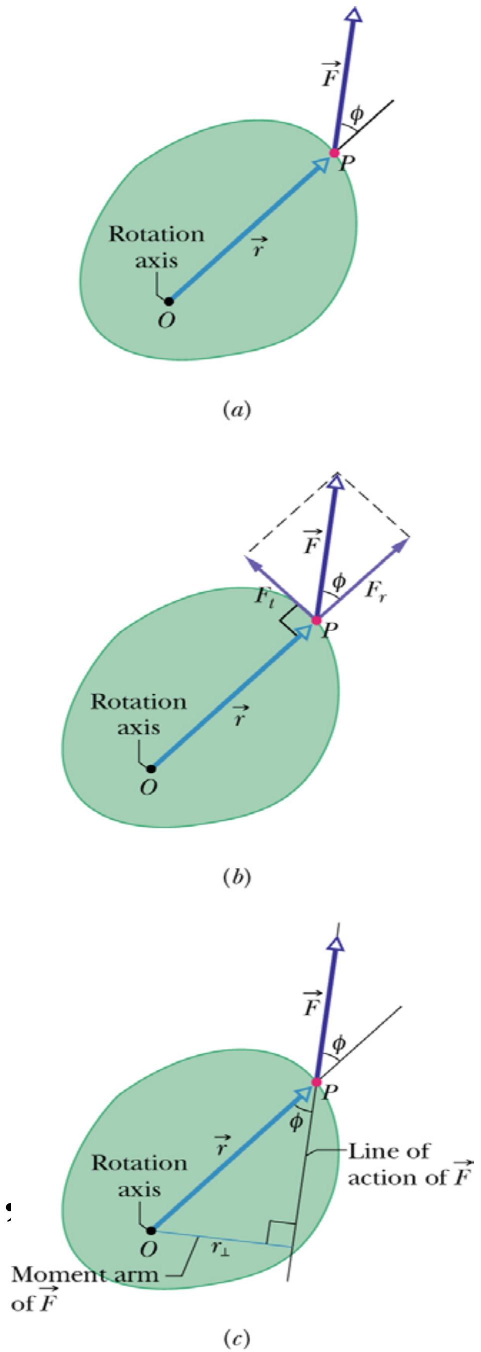
$$\tau = r F \sin \phi \qquad \vec{\tau} = \vec{r} \times \vec{F}$$

F is applied at point P.

r : distance from P to the rotation axis.

unit: N · m

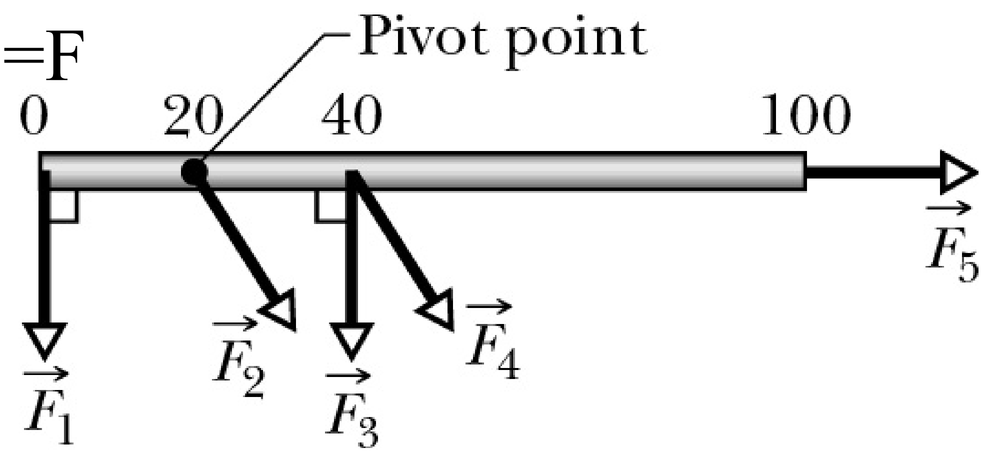
direction: "clockwise (CW) is negative" because the angle is decreasing



Checkpoint 10-6

The figure shows an overhead view of a meter stick that can pivot about the dot at the position marked 20 (for 20cm). All five horizontal forces on the stick have the same magnitude. Rank those forces according to the magnitude of the torque that they produce, greatest first.

$|F_1|=|F_2|=|F_3|=|F_4|=|F_5|=F$



$\vec{\tau} = \vec{r} \times \vec{F}$

Magnitude:

$\tau = r\,F \sin\phi$

F_1: $\tau_1 = r_1\,F_1 \sin\phi_1 = (20)F\sin(90^o) = 20F$ (CCW)

Checkpoint 10-6

The figure shows an overhead view of a meter stick that can pivot about the dot at the position marked 20 (for 20cm). All five horizontal forces on the stick have the same magnitude. Rank those forces according to the magnitude of the torque that they produce, greatest first.

$|F_1|=|F_2|=|F_3|=|F_4|=|F_5|=F$

$\vec{\tau} = \vec{r} \times \vec{F}$

Magnitude:

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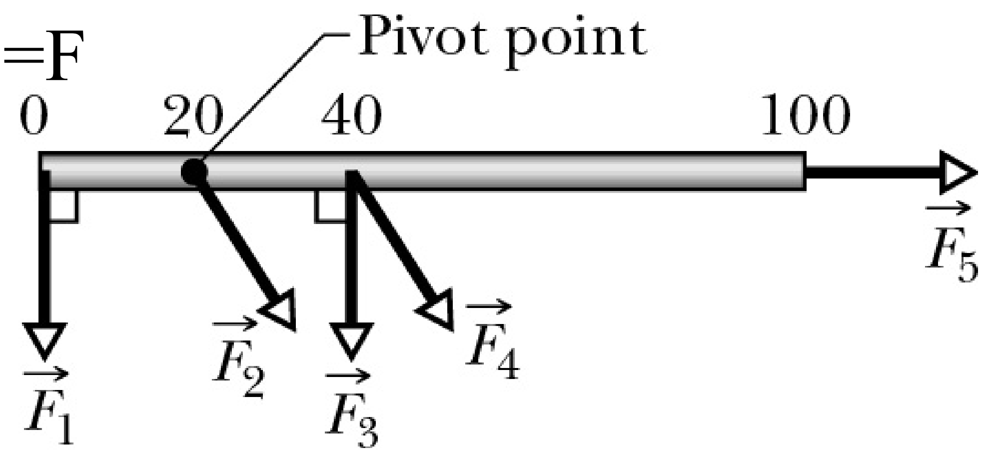


F_2: $\tau_2 = r_2\, F_2 \sin\phi_2 = (0)F\sin(60^o) = 0$ (no direction)

Checkpoint 10-6

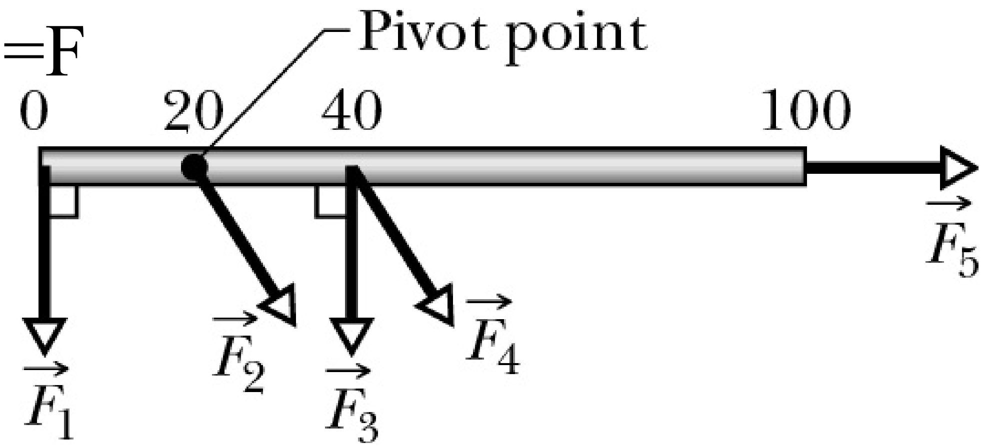
The figure shows an overhead view of a meter stick that can pivot about the dot at the position marked 20 (for 20cm). All five horizontal forces on the stick have the same magnitude. Rank those forces according to the magnitude of the torque that they produce, greatest first.

$|F_1|=|F_2|=|F_3|=|F_4|=|F_5|=F$

$\vec{\tau} = \vec{r} \times \vec{F}$

Magnitude:

$\tau = r F \sin \phi$



F_3: $\tau_3 = r_3 F_3 \sin \phi_3 = (20)F\sin(90^o) = 20F$ (CW)

Checkpoint 10-6

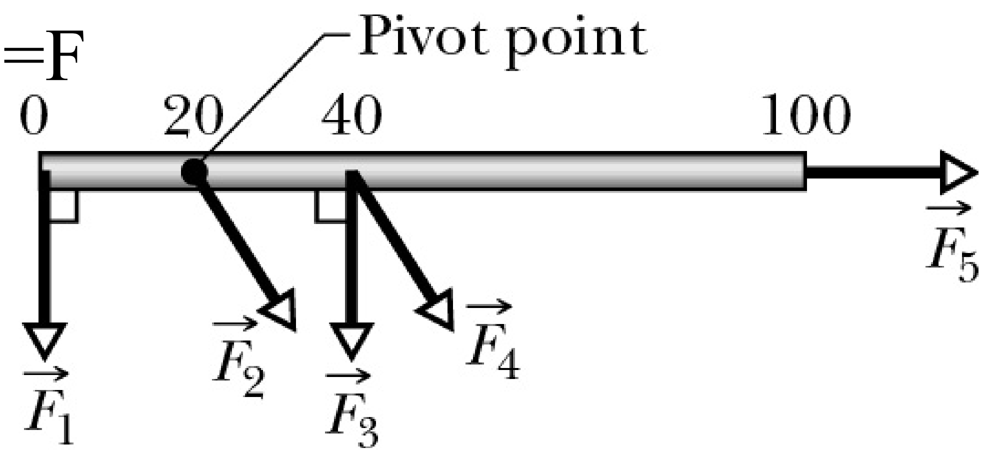
The figure shows an overhead view of a meter stick that can pivot about the dot at the position marked 20 (for 20cm). All five horizontal forces on the stick have the same magnitude. Rank those forces according to the magnitude of the torque that they produce, greatest first.

$|F_1|=|F_2|=|F_3|=|F_4|=|F_5|=F$

$\vec{\tau} = \vec{r} \times \vec{F}$

Magnitude:

$\tau = r\, F \sin \phi$



F_4: $\tau_4 = r_4\, F_4 \sin \phi_4 = (20)F\sin(60^o) = 17.3F$ (CW)

Checkpoint 10-6

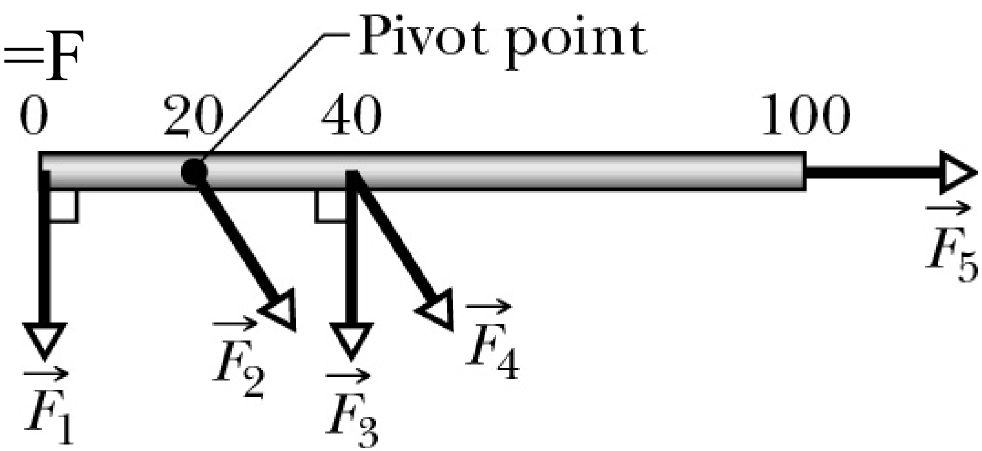
The figure shows an overhead view of a meter stick that can pivot about the dot at the position marked 20 (for 20cm). All five horizontal forces on the stick have the same magnitude. Rank those forces according to the magnitude of the torque that they produce, greatest first.

$|F_1|=|F_2|=|F_3|=|F_4|=|F_5|=F$

$\vec{\tau} = \vec{r} \times \vec{F}$

Magnitude:

$\tau = r\,F \sin\phi$



F_5: $\tau_5 = r_5\,F_5 \sin\phi_5 = (80)F\sin(0^o) = 0$ (no direction)

- **Newton's second law for rotation**

$$\vec{\tau}_{net} = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_i = I\vec{\alpha}$$

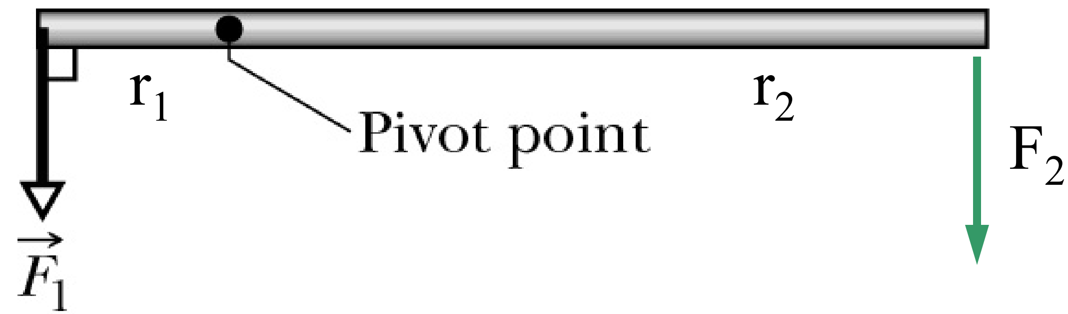
I: rotational inertia

α: angular acceleration

Compare to the linear equation: $\vec{F}_{net} = \sum_{i=1}^{n} \vec{F}_i = m\vec{a}$

Check point 10-7: Forces F_1 and F_2 are applied on a meter stick which is free to rotate around the pivot point. Only F_1 is shown. F_2 is perpendicular to the stick (in the same plane as F_1 and the stick) and is applied at the right end. If the stick is not to rotate, then

(a) what should be the direction of F_2?

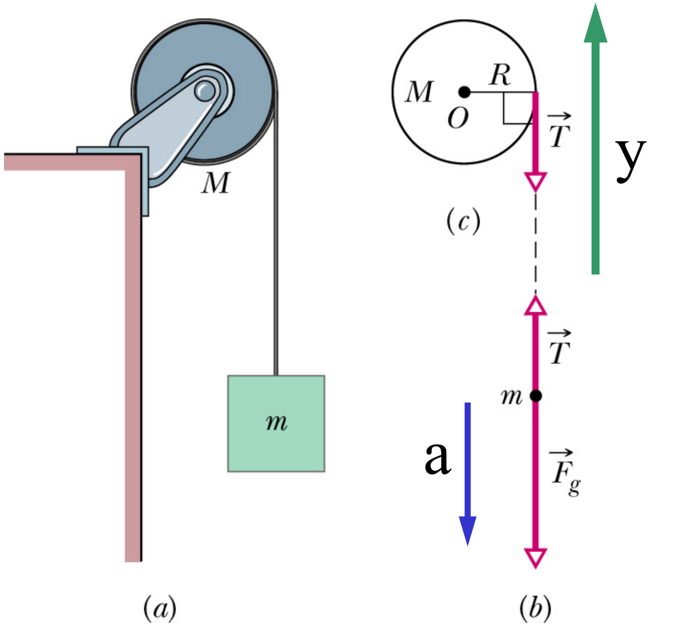


(b) Should F_2 be greater than, less than, or equal to F_1?

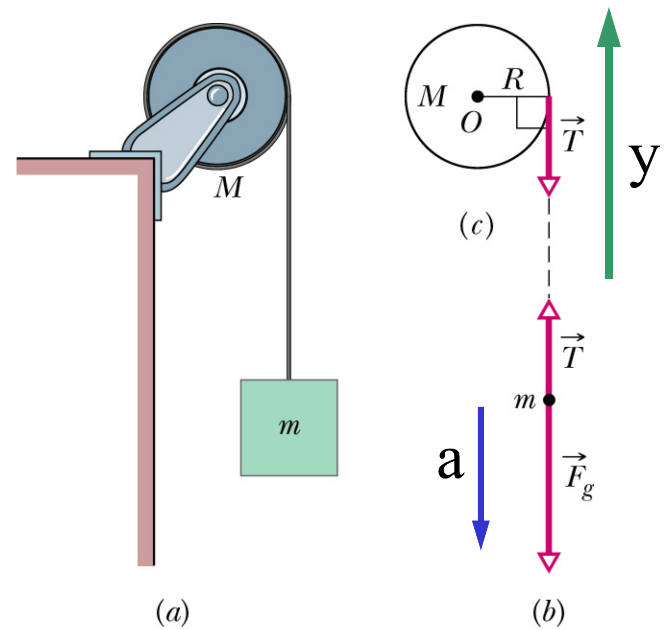
$r_2 > r_1$ => F_2 must be less that F_1 so the torques cancel

Sample Problem 10-8. This figure shows a uniform disk, with mass M=2.5kg and radius R = 20 cm, mounted on a fixed horizontal axle. A block with mass m = 1.2 kg hangs from a massless cord that is wrapped around the rim of the disk. Find the acceleration of the falling block, the angular acceleration of the disk, and the tension in the cord.

Note: R = 20 cm = 0.2m



Sample Problem 10-8. This figure shows a uniform disk, with mass M=2.5kg and radius R = 20 cm, mounted on a fixed horizontal axle. A block with mass m = 1.2 kg hangs from a massless cord that is wrapped around the rim of the disk. Find the acceleration of the falling block, the angular acceleration of the disk, and the tension in the cord.



The key points are following:

•For the block:

$$T - mg = ma \qquad (1)$$

•For the pulley: $\tau = I\,\alpha$

$$-RT = (1/2)MR^2\,\alpha \qquad (2)$$

•acceleration a of the block is equal to

a_t at the rim of the pulley

$$a = a_t = \alpha\,R \qquad (3)$$

Three equations and three unknowns: a, α, T, so we should be able to solve it.

Sample Problem 10-8. This figure shows a uniform disk, with mass M=2.5kg and radius R = 20 cm, mounted on a fixed horizontal axle. A block with mass m = 1.2 kg hangs from a massless cord that is wrapped around the rim of the disk. Find the acceleration of the falling block, the angular acceleration of the disk, and the tension in the cord.

Equations 2&3:

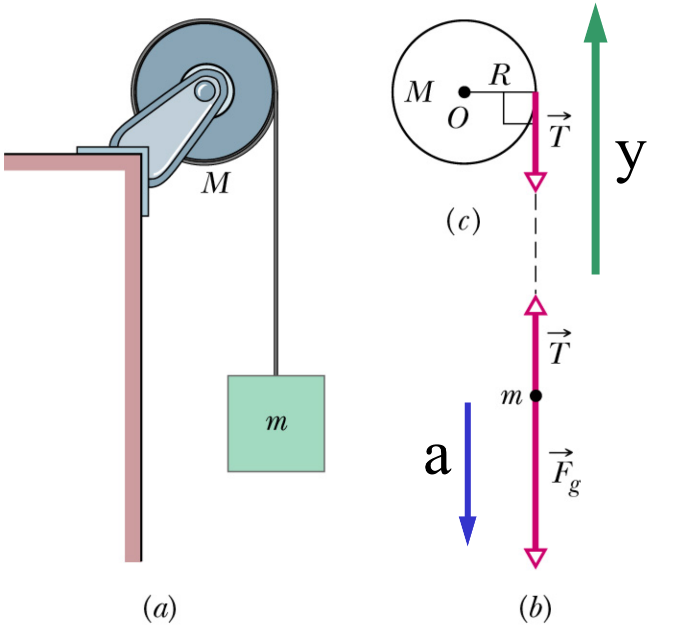
$T = -(1/2)Ma$

Equations 2&3 and 1:

$a = \alpha R = -(Ma)/2m - g$

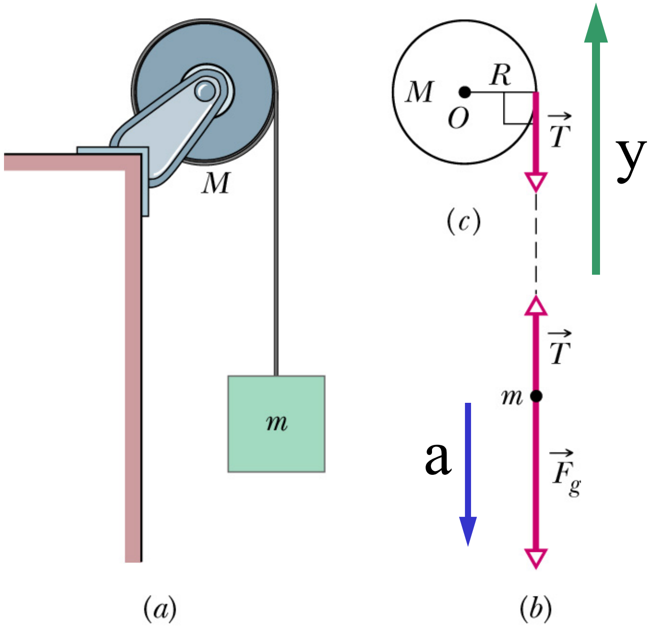
Equations 1,3 and 2:

$a = \alpha R = g - MR\,\alpha/2m$



$$\left(1+\frac{M}{2m}\right)a = -g \quad \Rightarrow \quad a = -\frac{2m}{(M+2m)}g = -4.8\text{m/s}^2$$

Sample Problem 10-8. This figure shows a uniform disk, with mass M=2.5kg and radius R = 20 cm, mounted on a fixed horizontal axle. A block with mass m = 1.2 kg hangs from a massless cord that is wrapped around the rim of the disk. Find the acceleration of the falling block, the angular acceleration of the disk, and the tension in the cord.



Then,

$T = -1/2Ma = 6.0N$

and,

$\alpha = a/R = -24\ rad/s^2$

Work and Rotational Kinetic Energy

- Work-kinetic energy theorem : $W = \Delta K = K_f - K_i$
  $\Delta K = \frac{1}{2} I \omega_f^2 - \frac{1}{2} I \omega_i^2$ (if there is only rotation)

- Work done $W = \int_{\theta_i}^{\theta_f} \tau \, d\theta$ (compare to $W = \int_{x_i}^{x_f} \vec{F} \cdot d\vec{x}$)

  if τ is constant, $W = \tau (\theta_f - \theta_i)$

- Power
  $P = dW/dt$
  $P = \tau \, d\theta/dt = \tau \, (d\theta/dt) = \tau \, \omega$ comparing to $P = F \, v$

translational motion	Quantity	Rotational motion
x	Position	θ
Δx	Displacement	$\Delta\theta$
$v = dx/dt$	Velocity	$\omega = d\theta/dt$
$a = dv/dt$	Acceleration	
m	Mass Inertia	I
$F = ma$	Newton's second law	$\vec{\tau} = \vec{r} \times \vec{F}$
$W = \int_{x_i}^{x_f} \vec{F} \cdot d\vec{x}$	Work	$W = \int_{\theta_i}^{\theta_f} \tau \, d\theta$
$K = \frac{1}{2} mv^2$	Kinetic energy	$K = \frac{1}{2} I\omega^2$
$P = \vec{F} \cdot \vec{v}$	Power (constant F or τ)	$P = \tau\omega$