Chapter 15: Oscillations

- Oscillations are motions that repeat themselves.

springs, pendulum, planets, molecular vibrations/rotations

- <u>Frequency f</u> : number of oscillations in 1 sec.

  unit: 1 hertz (Hz) = 1 oscillation per second

  = 1 s^{-1}

- <u>Period T</u> : the time for one complete oscillation. It is the inverse of frequency.

  $T = 1/f$ or $f = 1/T$

Simple Harmonic Motion

- Simple harmonic motion (SHM) is the oscillation in which the displacement x(t) is in the form of

$$x(t) = x_m \cos(\omega t + \phi)$$

x_m, ω and ϕ are constants

x_m: amplitude or maximum displacement, $x_{max} = A$

$\omega t + \phi$: phase; ϕ: phase constant

What is ω? since $x(t) = x(t + T)$

so $x_m \cos(\omega t + \phi) = x_m \cos(\omega(t+T) + \phi)$

thus $\omega(t+T) + \phi = (\omega t + \phi) + 2\pi \rightarrow \omega T = 2\pi$

therefore: $\omega = 2\pi/T = 2\pi f$

ω is called angular frequency (unit: rad/s)

- Displacement of SHM:

  $x(t) = x_m \cos(\omega t + \phi)$

- Velocity of SHM:

  $v(t) = dx(t)/dt = -\omega x_m \sin(\omega t + \phi)$

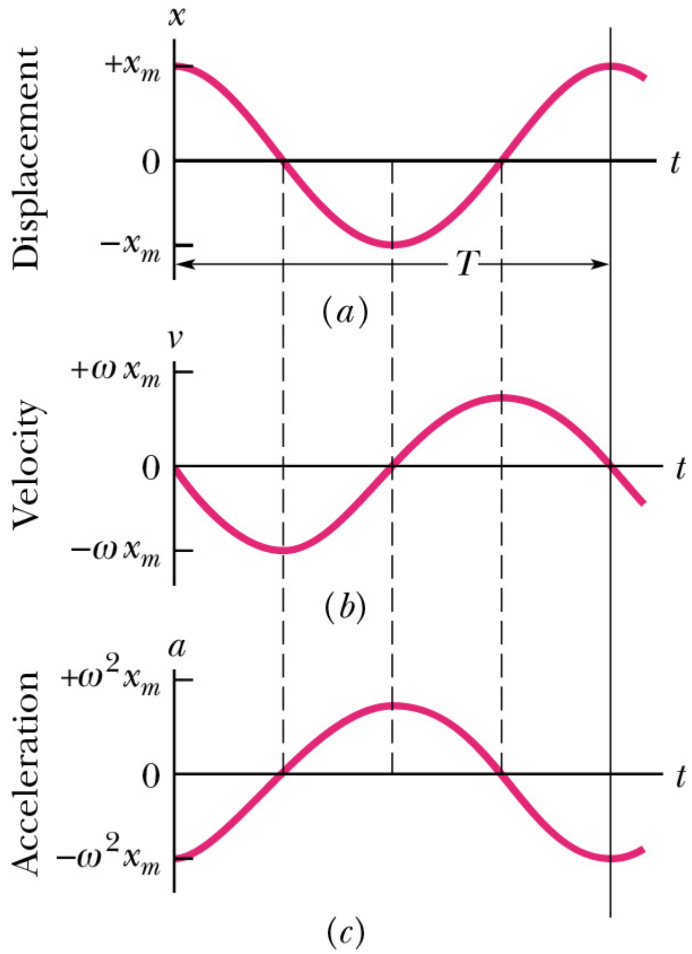
  velocity amplitude: $v_{max} = \omega x_m$

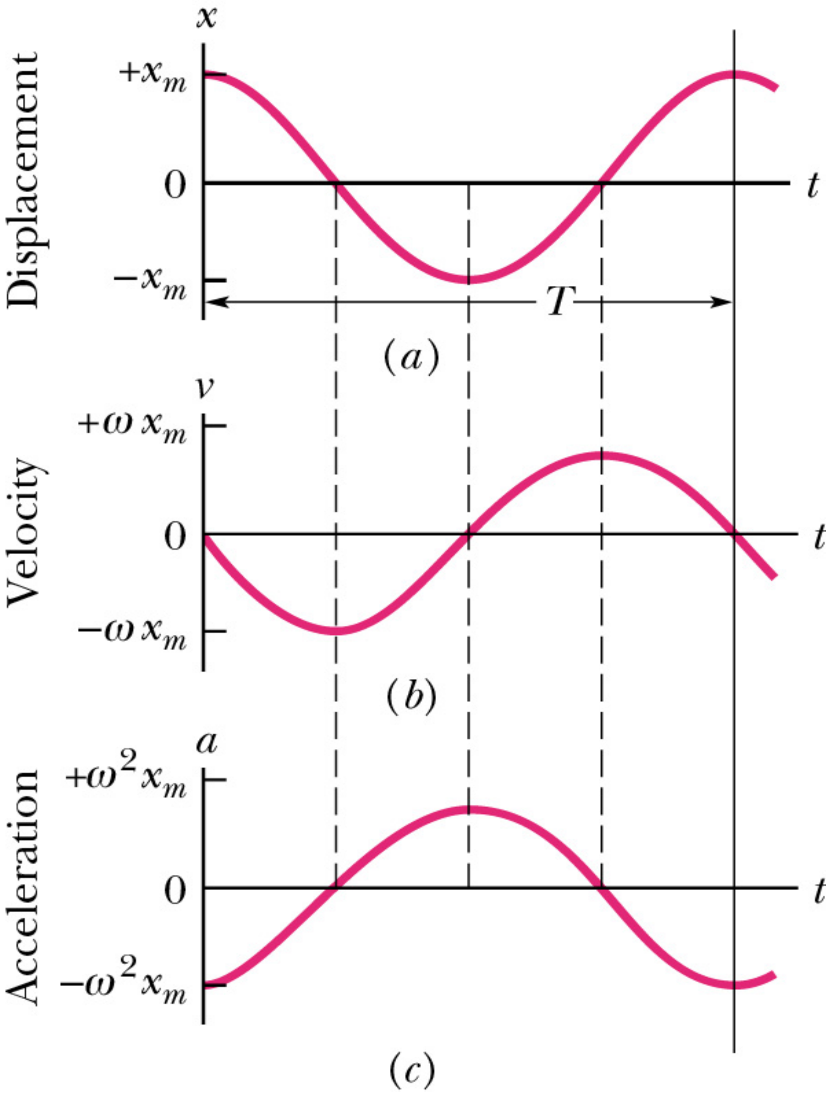
- Acceleration of SHM:

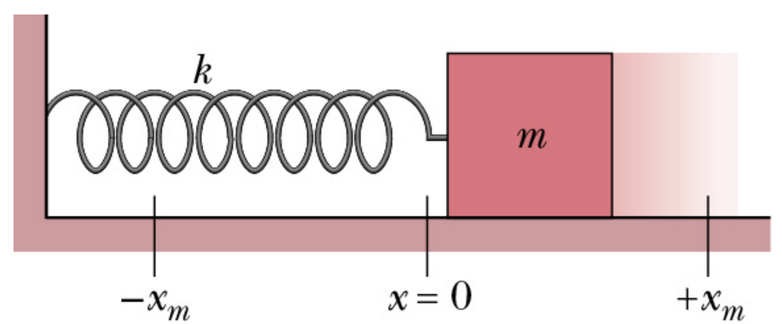
  $a(t) = dv(t)/dt = -\omega^2 x_m \cos(\omega t + \phi)$

  thus: $a(t) = -\omega^2 x(t)$

  maximum value of acceleration: $a_{max} = \omega^2 x_m$




Displacement
x
$+x_m$
0
t
$-x_m$
T
(a)
Velocity
v
$+\omega x_m$
0
t
$-\omega x_m$
(b)
Acceleration
a
$+\omega^2 x_m$
0
t
$-\omega^2 x_m$
(c)


k
m
x
$-x_m$
$x = 0$
$+x_m$

- The force law for SHM:

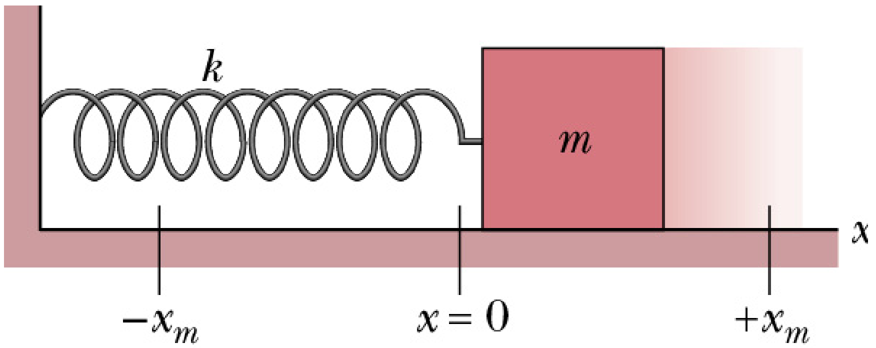
  since $a(t) = -\omega^2 x(t)$, $F = ma = -m\omega^2 x = -(m\omega^2)x$

  spring force fits this criteria: $F = -kx$
- Therefore, the block-spring system is a linear simple harmonic oscillator

$$k = m\omega^2$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$



A Quiz

Which of the following relationships between the force F on a particle and the particle's position x implies simple harmonic oscillation?

1) $F = -5x$
2) $F = -400x^2$
3) $F = 10x$
4) $F = 3x^2$
5) none of the above

A Quiz

$$F = ma(t) = m\,dv(t)/dt = -m\omega^2 x_m \cos(\omega t + \phi) = -(m\omega^2)x(t)$$

$$m\omega^2 = 5.0$$

Which of the following relationships between the force F on a particle and the particle's position x implies simple harmonic oscillation?

1) $F = -5x$
2) $F = -400x^2$
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4) $F = 3x^2$
5) none of the above

- # Energy in SHM

<u>Potential energy:</u>

$U(t) = \frac{1}{2} kx^2 = \frac{1}{2} kx_m^2 \cos^2(\omega t + \phi)$

<u>Kinetic energy:</u>

$K(t) = \frac{1}{2} mv^2 = \frac{1}{2} m\omega^2 x_m^2 \sin^2(\omega t + \phi)$

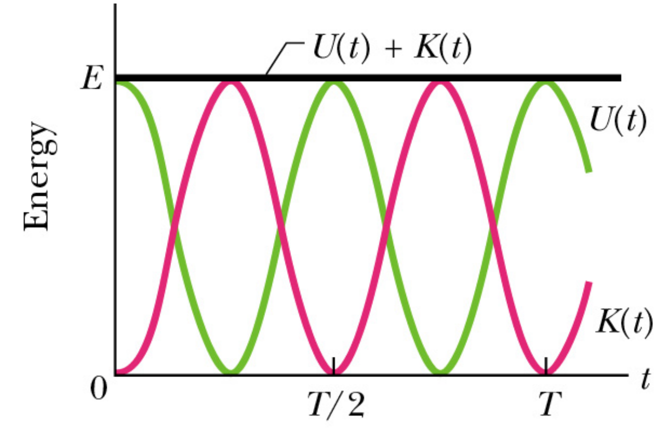
since $k/m = \omega^2$

$K(t) = \frac{1}{2} kx_m^2 \sin^2(\omega t + \phi)$

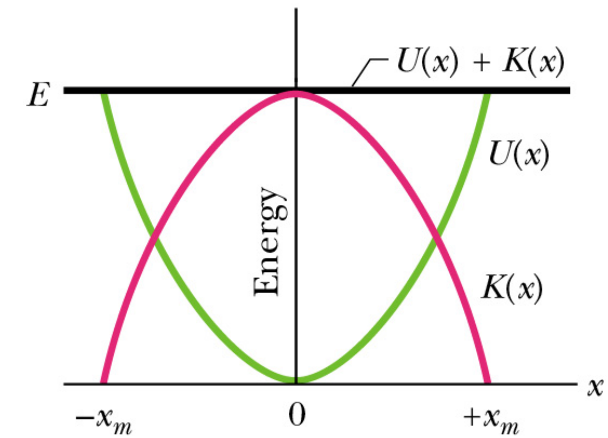
<u>Mechanical energy:</u>

$E = U + K = \frac{1}{2} kx_m^2[\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)] = \frac{1}{2} kx_m^2$

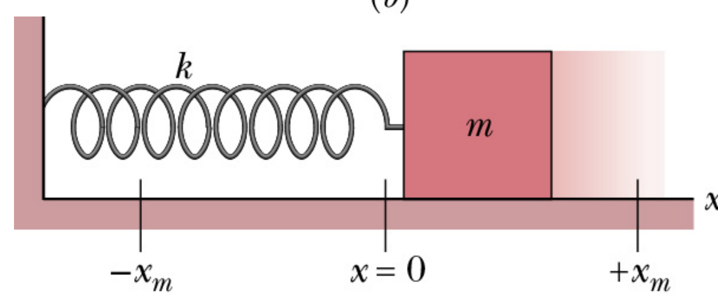
Mechanical energy is indeed a constant and is independent of time .



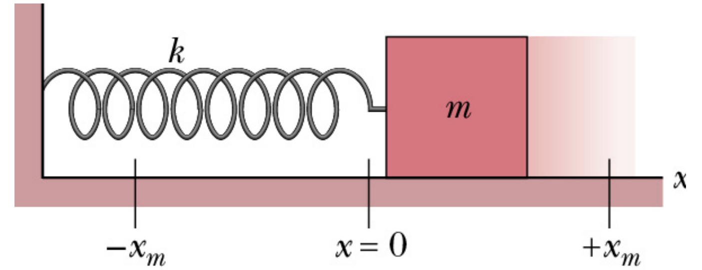
(a)



(b)



The block has a kinetic energy of 3.0J and the spring has an elastic potential energy of 2.0J when the block is at x = +2.0 m.



A) What is the kinetic energy when the block is at x = 0?

$$E = U(x,t) + K(x,t) = 5.0J = \tfrac{1}{2} mv^2 + U(0,t)$$

2.0 3.0 0

B) What is the potential energy when the block is at x = 0?

$$U(x,t) = \tfrac{1}{2} kx^2 = \tfrac{1}{2} kx_m^2 \cos^2(\omega t + \phi) = 0$$

C) What is the potential energy when the block is at x = –2.0m?

$$U(x,t) = U(-x,t') = 2.0J$$

D) What is the potential energy when the block is at $x = -x_m$?

$$E = U(t) + K(t) = \tfrac{1}{2} kx_m^2[\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)] = \tfrac{1}{2} kx_m^2 = 5.0J$$

0

An angular simple harmonic oscillator

- Torsion(twisting) pendulum

  restoring torque:

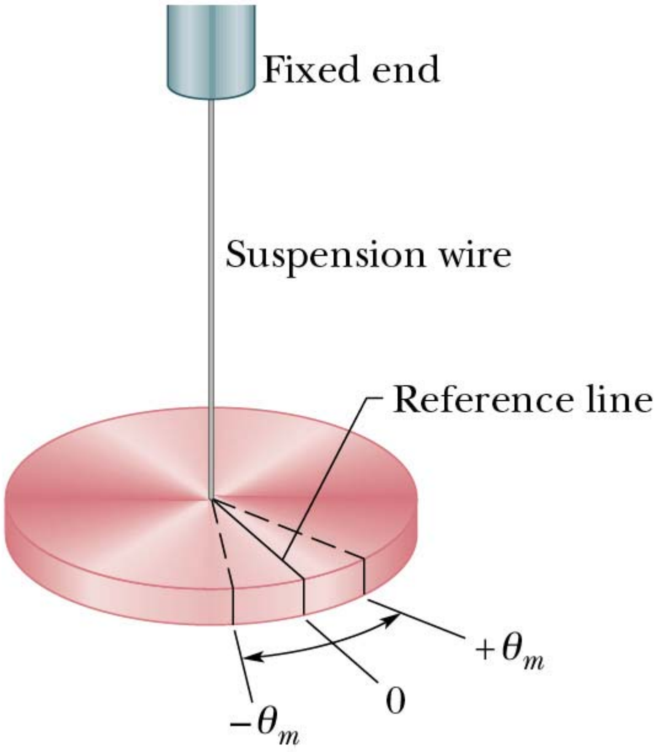
  $\tau = -\kappa\theta$

  compare to $F = -kx$ and

  $$T = 2\pi\sqrt{\frac{m}{k}}$$

  we have for angular SHM:

  $$T = 2\pi\sqrt{\frac{I}{\kappa}}$$



Pendulum

- **The simple pendulum**

restoring torque:

$\tau = -L(mg \sin\theta)$

"–" indicates that τ acts to reduce θ.

For small θ, $\sin\theta \sim \theta$

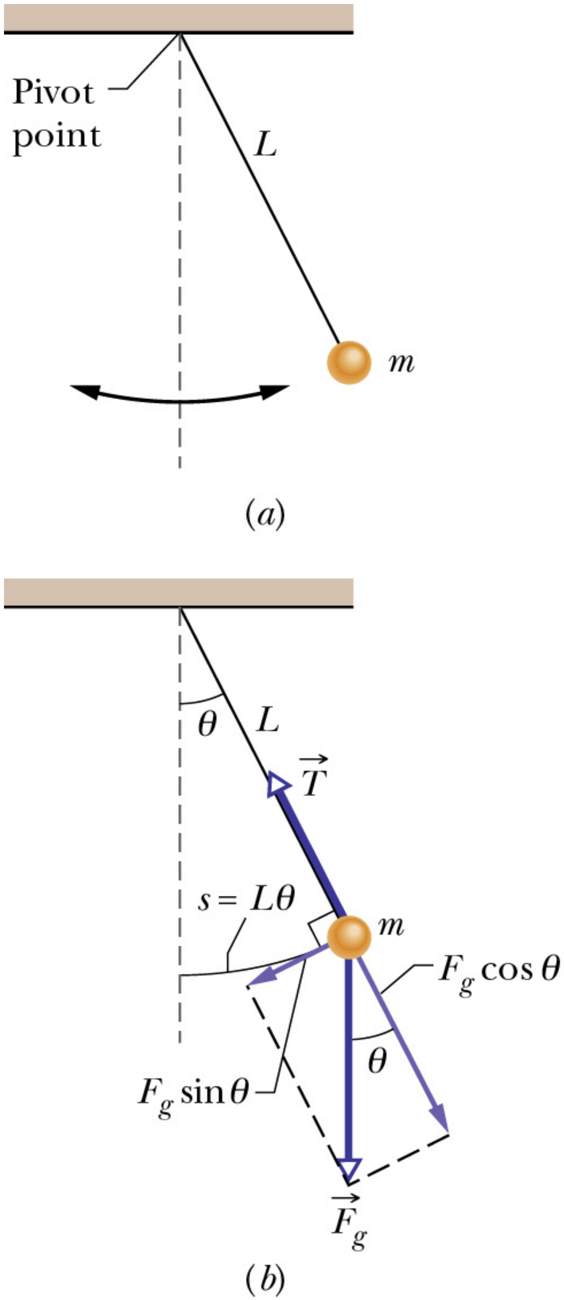
thus: $\tau = -Lmg\theta$

so: $\kappa = mgL$,

$$T = 2\pi\sqrt{\frac{I}{\kappa}} = 2\pi\sqrt{\frac{I}{mgL}} = 2\pi\sqrt{\frac{mL^2}{mgL}} = 2\pi\sqrt{\frac{L}{g}}$$

This can be used to measure g

$g = (2\pi/T)^2 L$

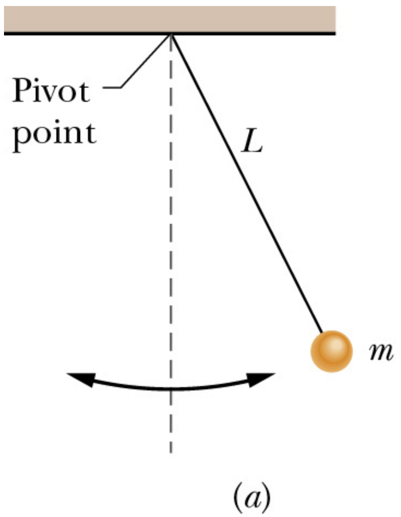


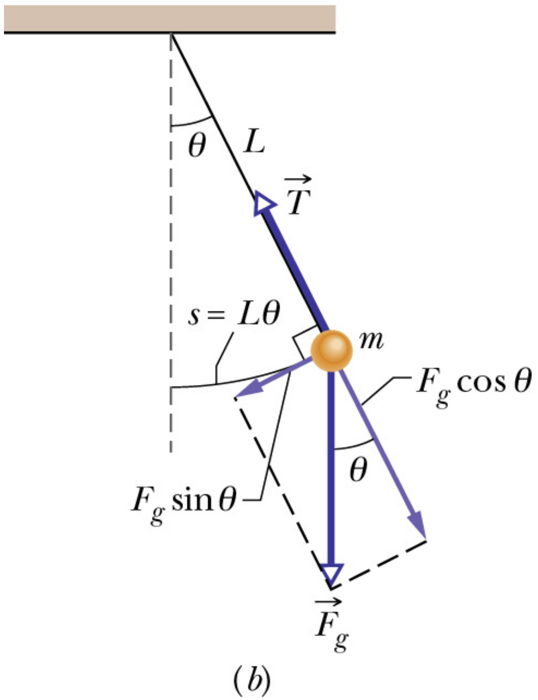
Pendulum

- **The simple pendulum**
  restoring force:
  F = – mg sinθ = – mg θ = – mg x/L
  when θ is small: sin θ = θ
  "–" indicates that F acts to reduce θ.
  so: k = mg/L

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{mg/L}} == 2\pi\sqrt{\frac{L}{g}}$$

This can be used to measure g
$g = (2\pi/T)^2 L$





- **The physical pendulum** (real pendulum with arbitrary shape)

  h: distance from pivot point O to the center of mass.

  Restoring torque:

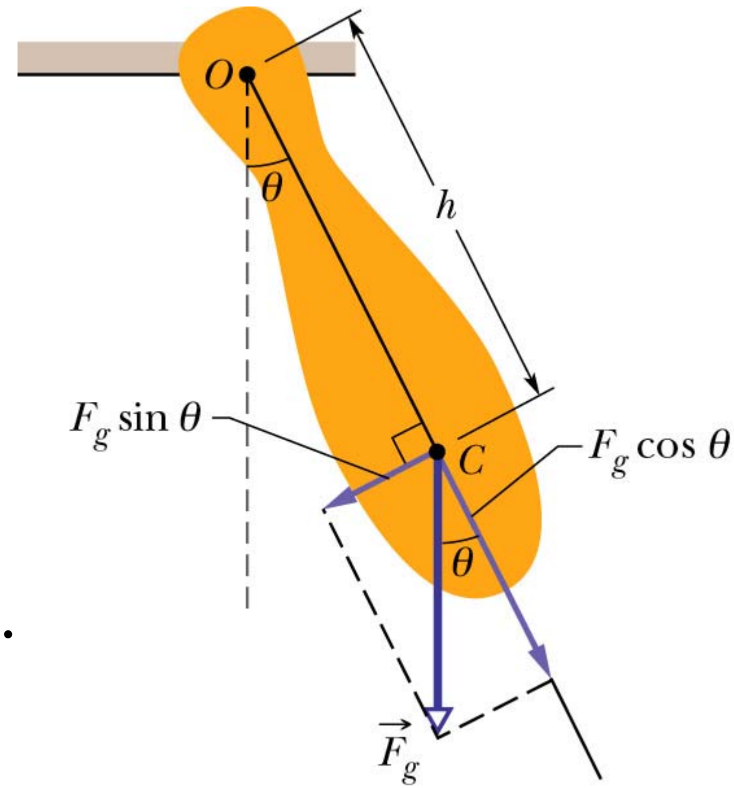
  $\tau = -h\,(mg\sin\theta) \sim (mgh)\theta$ (if θ is small)

  $\kappa = mgh$

  therefore:

$$T = 2\pi\sqrt{\frac{I}{\kappa}} = 2\pi\sqrt{\frac{I}{mgh}}$$

I: rotational inertia with respect to the rotation axis thru the pivot.



Simple harmonic motion and circular motion

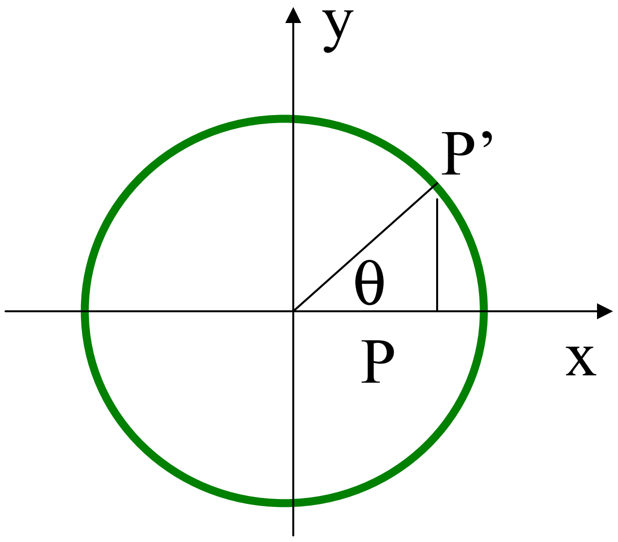
- Circular motion of point P':

  angular velocity: ω

  $\theta = \omega t + \phi$

  P is the projection of P'on x-axis:

  $x(t) = x_m \cos(\omega t + \phi)$ SHM

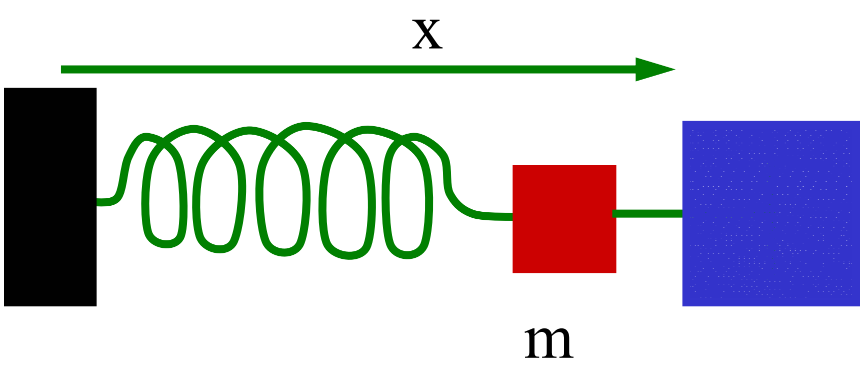


- P': uniform circular motion

  P: simple harmonic motion

Damped simple harmonic motion

- When the motion of an oscillator is reduced by an external force, the oscillator or its motion is said to be damped.
- The amplitude and the mechanical energy of the damped motion will decrease exponentially with time.



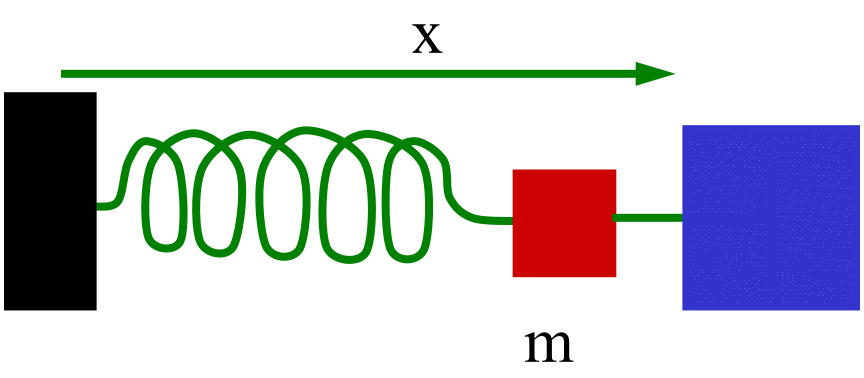
Damped simple harmonic motion

Spring Force $F_s(x,t) = -kx$

Damping Force $F_d(x,t) = -b\,v(t) = -b\left(\frac{dx}{dt}\right)$

Response (Newton's Second Law)

$$\sum_i F_i(x,t) = ma = m\left(\frac{d^2x}{dt^2}\right) \Rightarrow -kx - b\left(\frac{dx}{dt}\right) = m\left(\frac{d^2x}{dt^2}\right)$$



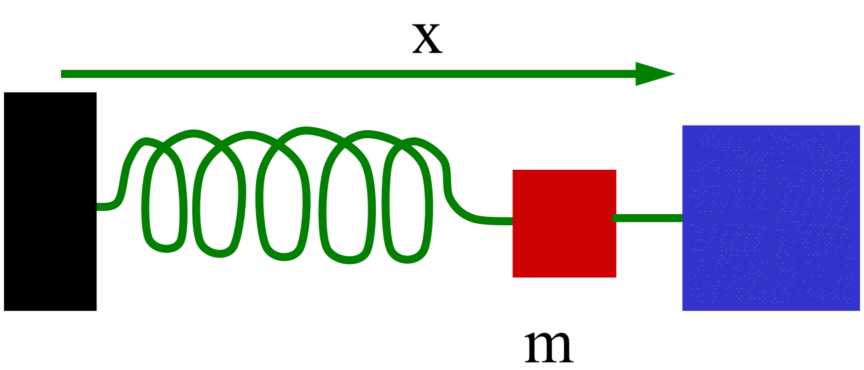
Damped simple harmonic motion

Response (Newton's Second Law)

Rearranging: $$m\left(\frac{d^2x}{dt^2}\right)+b\left(\frac{dx}{dt}\right)+kx=0$$

$$x(t)=x_m e^{-bt/2m}\cos(\omega' t+\phi)$$

where, $$\omega'=\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}$$



Forced Oscillation and Resonance

- Free oscillation and forced oscillation
- For a simple pendulum, the natural frequency $\omega_0 = \sqrt{\frac{g}{l}}$

- Now, apply an external force: $F = F_m \cos(\omega_d t)$

  ω_d, driving frequency

  x_m depend on ω_0 and ω_d,

  when $\omega_d = \omega_0$, x_m is about the largest

  this is called **resonance**.

  examples : push a child on a swing, air craft design, earthquake